THEOREM IV. *In the complete figure 1500 such conics exist.*

PROOF: The total number of intersections of the four p-lines through one fixed P-point with the four through another is 16. The total number of combinations of two points possible with 15 is 15.14/2. Hence the total number of intersections is 16.15.14/2 which is 1680. It was shown in Theorems I and III that when the intersection was in a P-point the result was a straight line. Hence 180 of these intersections form straight lines and 1500 form conics.

THEOREM V. *The 1500 conics are not all distinct, but 240 of them reduce to 80 distinct conics.*

PROOF: The proof of this theorem follows immediately from Steiner's Theorem and Kirkman's extension of Steiner's Theorem[1] which gives 80 points, in each of which there are three intersections of p-lines.

THEOREM VI. *Of the 105 combinations of two fixed points which serve as centers of projective pencils generating the 1500 conics, 15 combinations serve as centers for 12 conics each, 30 combinations for 14 conics each, and 60 combinations for 15 conics each.*

PROOF: Let the points 1, 2, 3, 4, 5, 6 be the six T-points. Then the opposite sides of the hexagon are 12 and 45, 23 and 56, and 34 and 61. The intersection of these opposite sides gives the three P-points on a p-line and we may choose the points in any order and select the numbers from the same position that we have above and still obtain a Pascal line.[2] Let us take any two orders—and it matters not which two we take, since we may interchange the numbers in any way so long as we draw from the proper positions—say $1\underline{23}\underline{54}\underline{6}$ and $1\underline{45}\underline{26}\underline{3}$. In the first, 23,46 will be a fixed P-point and the other eight P-points lying on the four lines through this point necessarily are:

12,45	13,54	12,56	13,56
61,35	25,61	14,35	14,52

Similarly 45,63 is the fixed P-point in the other order and the other eight corresponding points are:

14,26	15,26	14,23	15,23
13,52	13,42	16,52	16,24

We may classify these sets into three groups. In the first group both fixed P-points in the set of two may be determined by the same four numbers. Four points on the conic determine three P-points and five points may have five combinations of four points each; therefore, this group contains 15 sets and we may show as above that each scheme has four moving P-points in common. In

[1] Salmon, *Conic Sections*, 1924 Edition, p. 380.
[2] Salmon, loc. cit., p. 379.

the second group we may draw the same numbers from one of the underscored positions in each scheme. In this case the two fixed P-points in a set lie on the same line through two fixed T-points. There are ten of these lines and three fixed P-points lie on each line; hence, there are 30 sets in this group and we may show as above that each set has two moving P-points common to each scheme. The third group may contain all other combinations and hence 60 sets. Then, since there are 16 intersections in all between the 4 p-lines from each of any two fixed points, and those which ocur in P-points give straight lines, there must be 15 sets which generate 12 conics each and 30 sets, 14 conics each. Since each set must contain at least one straight line and there are 60 sets and 60 straight lines yet unused, each of these 60 sets must each contain one straight line and hence 15 conics.

Corollary. It may easily be shown by a study of combinations that the four straight lines in a set, mentioned above, are the lines joining the four fixed T-points, which determine the two fixed P-points in the set, to the fifth point which is not so used and hence the four lines are always distinct and intersect in one of the four T-points. Also when all five fixed T-points are used to determine the two fixed P-points in the set, that the two points which are so used only once determine the only straight line in the set. Hence, the two straight lines in each of the 30 sets above are never distinct, but two intersections describe the same line.

THEOREM VII. *The loci of the 16 intersections of the four p-lines through each of any two fixed P-points in 90 sets out of the 105 intersect eight by eight*[1] *in two of the fixed T-points and four by four in four other points, two of which lie on a line joining one of the two T-points above with one of the two fixed P-points. The other two lie on a line determined by the second fixed T-point above and the second fixed P-point. They also intersect four by four in each of the other three fixed T-points.*

In the remaining 15 out of the 105 sets the 16 loci all intersect in one fixed T-point and four by four in each of the other four fixed T-points, and do not intersect four by four in other points.

PROOF: Consider first any two fixed P-points determined by all five of the fixed T-points, say 63,45 and 65,42 and the eight p-lines which pass through these four by four determining the 16 intersections in the set. They are:

$$1\begin{cases}63,45\\61,52,\\32,14\end{cases} \quad 2\begin{cases}63,45\\61,24,\\15,23\end{cases} \quad 3\begin{cases}63,45\\14,62,\\25,13\end{cases} \quad 4\begin{cases}63,45\\13,24,\\26,15\end{cases}$$

$$5\begin{cases}65,42\\15,32,\\63,41\end{cases} \quad 6\begin{cases}65,42\\12,63,\\51,34\end{cases} \quad 7\begin{cases}65,42\\61,34,\\53,12\end{cases} \quad 8\begin{cases}65,42\\52,41.\\61,32\end{cases}$$

[1] Note: In 30 sets the eight loci are not all distinct but the locus of two intersections coincide upon the line determined by the two points. See Cor. Theorem VI.

When point 1 coincides with point 2 the intersections of lines 5 and 6 with 1, 2, 3, and 4 all coincide upon point 2. Hence, eight loci must pass through this point. Likewise when point 1 coincides with point 3 the intersections of lines 1 and 2 with 5, 6, 7, and 8 coincide upon point 3 and eight loci must pass through point 3. These are the two points used once only in determining the two fixed P-points in the set. Now, let point 1 coincide with a fixed point which is used twice, say point 4, and the only intersections coinciding with point 4 are the lines 2 and 4 with 6 and 7; hence four loci intersect at point 4 and likewise we may show that four intersect at 5 and 6 each. Now, let point 1 coincide with point 3 again and the lines 5, 6, 7, and 8 coincide and are determined by the fixed P-point 65,42 and point 3, but lines 3 and 4 cut this line in two points other than point 3. Hence the loci intersect 4 by 4 on two points on that line other than the point 3. Similarly, if point 1 coincides with point 2 the lines 1, 2, 3, and 4 coincide and are determined by the fixed P-point 63,45 and point 2 and lines 6 and 7 cut this line in two points other than the point 2. Hence upon this line there are two points in which the loci intersect 4 by 4. Since we may choose the points 1, 2, 3, 4, 5, 6 without respect to order we may interchange points as we please in the above scheme with no restriction except that the same change be carried through one p-line. But even so, only two cases are possible. First, in 90 sets out of the 105 the two fixed P-points in each set will be determined by all five fixed T-points, two being used once only and three twice each as above and in these the interchange of points will not disturb the above reasoning. In the second case there are 15 sets (see theorem VI) in which the two fixed P-points are determined by only 4 T-points, each used twice and one point not used at all. We now examine the intersections in this case. We may take as our eight lines:

$$1\begin{cases}63,45\\61,52,\\32,14\end{cases}\qquad 2\begin{cases}63,45\\61,24,\\15,23\end{cases}\qquad 3\begin{cases}63,45\\14,63,\\25,12\end{cases}\qquad 4\begin{cases}63,45\\13,24,\\26,15\end{cases}$$

$$5\begin{cases}65,43\\16,24,\\52,31\end{cases}\qquad 6\begin{cases}65,43\\15,24,\\62,31\end{cases}\qquad 7\begin{cases}65,43\\15,23,\\62,41\end{cases}\qquad 8\begin{cases}65,43\\16,23.\\52,41\end{cases}$$

Now, let point 1 coincide with point 2 and the lines 1, 2, 3, and 4 coincide and are determined by the fixed P-point 63,45 and the point 2. Hence all 16 loci intersect in point 2, and since none of the first four lines intersect the last four except in the point 2, under this arrangement the four points above on these two special lines in which the loci intersected 4 by 4 do not exist in this case. We may show as before by letting point 1 coincide with 3, 4, 5, and 6 respectively that the loci intersect 4 by 4 in these points.

Interchanging of points in this case simply gives our 15 sets, or changes the scheme back to the first case. Hence the theorem is established.

THEOREM VIII. *On the line joining a fixed P-point to a fixed T-point, which is not used in determining the P-point, there lie two points through each of which*

twelve distinct conics pass and eight points through each of which eight conics pass. There are fifteen of these lines passing by threes through each fixed point on the conic.

PROOF: Due to symmetry we need to investigate only one line. We shall take the line joining the P-point 36,25 to point 4 on the conic. Since four p-lines coincide upon this line when point 1 coincides with point 4, there must be four Steiner points on this line. They are:[1]

$$S_1\begin{cases}36,25,41\\45,61,23,\\21,34,56\end{cases}\qquad S_2\begin{cases}36,25,41\\15,64,23,\\42,31,56\end{cases}\qquad S_3\begin{cases}25,36,41\\31,54,26,\\46,12,35\end{cases}\qquad S_4\begin{cases}25,36,41\\61,24,35.\\43,15,26\end{cases}$$

When point 1 coincides with point 4, S_1 coincides with S_2 and S_3 with S_4. There are also twelve Kirkman points on this line, but an entirely similar investigation shows that two of these coincide with S_1 and two with S_3 and the remaining eight coincide with point 4 on the conic. When two points on the conic coincide the number of Pascal lines reduces to thirty, the number of permutations of six points on a curve taking two at a time and two alike. By making a study of these similar to the proof of Theorem VII, it may easily be shown that these thirty lines are not all distinct but six of them coincide by twos with the three lines connecting point 4 to the P-points determined by points 2, 3, 5, and 6. Twelve others coincide by threes on the four lines joining point 4 to points 2, 3, 5, and 6. The other twelve are distinct and since the three columns of S_1 and S_2 become identical two of these intersect at S_1 and similarly two at S_3. Since all the Steiner and Kirkman points on the line in question are located, the remaining eight of the twelve lines must intersect this line in eight points. We now have three lines intersecting at S_1, one containing four p-lines and two containing two p-lines each, hence there are twenty intersections of p-lines at this point. But there are also two Steiner and two Kirkman points there (see theorem V); hence, there are twelve distinct conics intersecting in this point and likewise twelve at S_3. In each of the other eight points two lines intersect, one containing four p-lines and one containing two p-lines. Hence at each of these points eight conics intersect. In Theorem VII it was shown that fifteen of these lines exist, three passing through each fixed point on the conic.

It is of interest to note that the last eight of the twelve lines mentioned above intersect by twos in four S-points lying two on each of the other two lines through point 4 on which four p-lines coincide.

THEOREM IX. *If we should relate six tangents to a conic as we have six points on a conic, then a line joining two intersections of four of the five fixed tangents determines a point on the fifth fixed tangent at which two lines intersect each tangent to twelve distinct conics and eight lines each tangent to eight conics. There are fifteen of these points lying by threes on each of the five tangents.*

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[1] The notation used here is interpreted thus: the first and second columns, the first and third columns, and the second and third columns give the 3 Pascal lines which meet in a Steiner point.

Since this is the exact dual of Theorem VIII, we shall not give a separate proof.

A few correspondences in the above investigation suggest certain theorems on the complete pentagon. Consider the three p-lines intersecting in a Steiner point,

$$\begin{cases} 36,25,41 \\ 45,61,23 \\ 12,34,56 \end{cases}$$

When 1 coincides with 4, four p-lines coincide upon the position of the p-line determined by the first two columns. This line is determined by the point 36,25 and point 4. Likewise the first and third columns determine a p-line. When 1 coincides with 6 four p-lines coincide and their position is fixed by the point 45,23 and the point 6. The second and third columns determine a similar set through the points 34,56 and 2. Although these three groups of four p-lines do not take these positions at the same time, but only as 1 coincides with 4, 6 and 2 respectively, the fact that the three lines that give rise to these positions meet in a Steiner point suggests that these three lines might be concurrent. Also since four p-lines coincide on each position it seems that there should be four of these points on each line. Hence, the following theorem is suggested.

THEOREM X. *The fifteen lines i formed by joining each of the five vertices of a pentagon in turn to the three points determined by lines joining the other four vertices, intersect in threes in twenty points G in addition to the vertices and four G-points lie on each i-line.*

PROOF: We shall use five T-points as the vertices in order to keep the notation uniform. Then in the two triangles whose sides are respectively (32), (21), (15), and (45), [(45,13)–(25,34)], (34) the corresponding sides meet in three points (23,45), (34,51) and the intersection of [(13,45)–(34,52)] with (2,1). But by Pascal's theorem we may show that the line joining the first two of these meets t_4 in the third, hence the three lines joining corresponding vertices meet in a point. But these are the three i-lines,

$$\begin{array}{l} (31,45)-2 \\ (23,51)-4. \\ (25,34)-1 \end{array}$$

The notation shows that if 3 and 1 be exchanged or 4 and 5 or both these changes be made at the same time, that the first line is unchanged. Hence four of these G-points lie on each i-line and since the numbers may be assigned to the points without respect to order the proof is complete.

We shall now state the dual of theorem X for reference in the following discussion.

THEOREM XI. *The fifteen points I formed by the intersection of each of the five sides of a pentagon in turn with three lines determined by joining the intersections*

of the other four sides, lie in threes on twenty lines g in addition to the sides and four g-lines pass through each I point.

Dual Properties: If we let the five fixed T-points be the vertices of the pentagon of theorem X and the corresponding t-lines the sides of the pentagon of theorem XI, we may develop an interesting cycle of pole and polar relationships among the lines through a T-point and the points on these lines.

The fifteen i-lines are now determined by joining in the proper way the fifteen fixed P-points to the five fixed T-points. The three P-points determined by a given four T-points determine the three i-lines through the fifth T-point, say T_n. But these three P-points are the vertices of a self-polar triangle; hence, the sides of this triangle determine the three I-points on t_n. The fifteen P-points determine 105 lines, thirty of these are formed each by a given two T-points, fifteen are the sides of the five sets of self-polar triangles and are formed each by a given four T-points. The remaining sixty are determined by using all five T-points and are therefore of the form (13,52)–(45,23). We shall call these the k-lines and their poles the K-points. It was shown in theorem VIII that these k-lines intersect by twos in thirty S-points lying by twos on each i-line. Hence, the s-lines are the lines of theorem IX which are each tangent to twelve conics. We may determine a certain set of six S-points using the method of theorem VIII from the six Steiner points:

$$S_1\begin{cases}36,25,41\\45,61,23,\\12,34,56\end{cases} \qquad S_2\begin{cases}36,45,12\\52,61,43,\\14,23,56\end{cases} \qquad S_3\begin{cases}25,36,41\\31,54,26,\\46,12,35\end{cases}$$

$$S_4\begin{cases}36,45,12\\25,31,46,\\14,26,35\end{cases} \qquad S_5\begin{cases}45,23,16\\26,41,35,\\13,65,42\end{cases} \qquad S_6\begin{cases}13,45,26\\56,32,14.\\24,16,35\end{cases}$$

When the moving vertex coincides with a fixed T-point, say point 4, S_1 and S_3 lie on i_1; S_2 and S_6 lie on i_3; and S_4 and S_5 lie on i_2 where the three i-lines pass through point 4. (See Figure 2.) Steiner has shown[1] that the three groups of two, S_1, S_2 and S_3, S_4 and S_5, S_6 are each divided harmonically by the conic. Hence S_1 lies on s_2 and S_2 lies on s_1.[2] Now, through S_1 pass k_1, k_2, i_1, s_2 and twelve conics. (See Figure 2.) Similarly, on s_1 which is tangent to twelve conics lie K_1, K_2, I_1, and S_2. A similar relationship exists among the two S-points in each of the other two groups, tying the six S-points, the six s-lines, the three I-points and the three i-lines together in an interesting figure. There are five of these sets in the complete figure. Figure two shows a cycle of these relationships.

Since the six points on the conic may be grouped into six sets of five points: theorem X and a generalization of the discussion following theorem XI give,

THEOREM XII. *The forty-five Pascal points lie by twos on 360 lines k, sixteen through each Pascal point, which intersect by twos in 180 points S. The S-points*

[1] Salmon, loc. cit., p. 380.

[2] Salmon, loc. cit., p. 148.

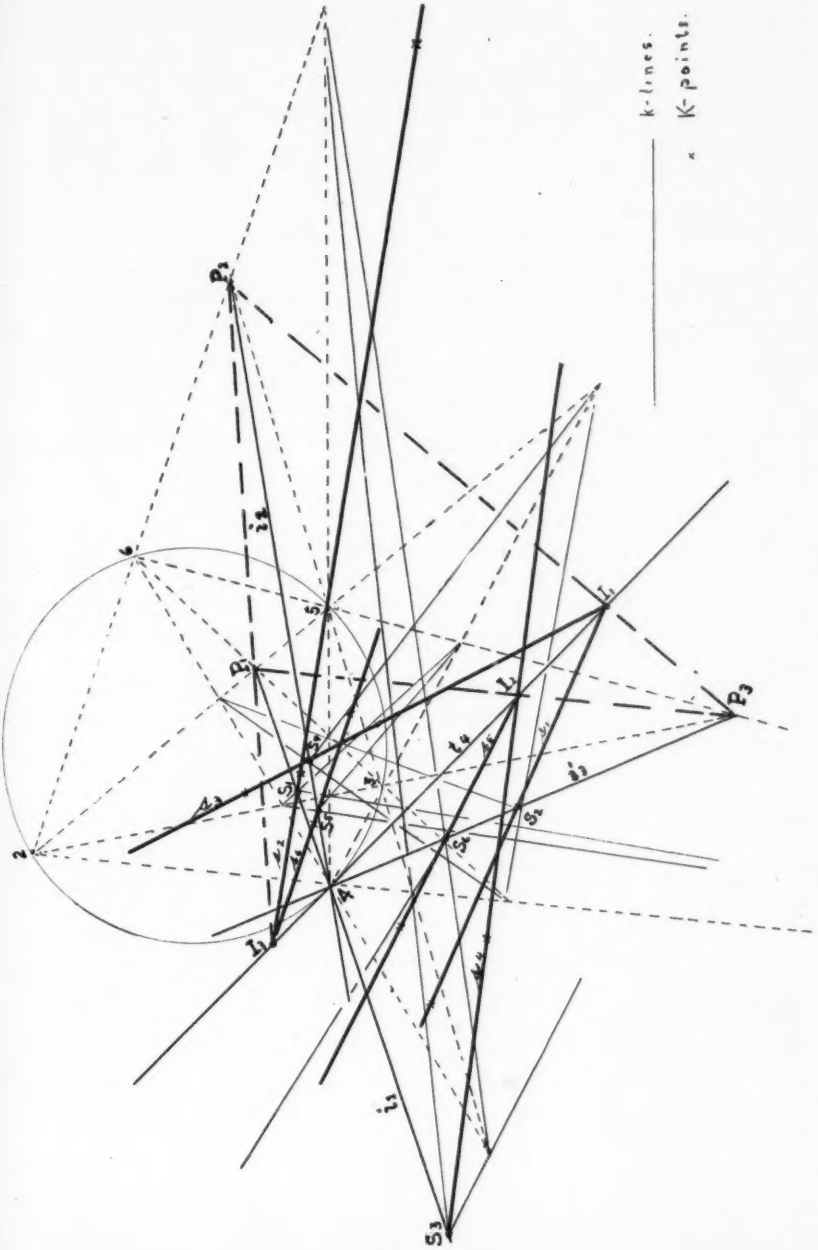

FIG. 2

lie by twos on 90 lines i. The i-lines intersect in groups of fifteen in each vertex of the hexagon and by twos in each of the forty-five Pascal points and by threes in 120 points G which lie in sets of four on each line. The S-points also lie in 90 sets of two which are harmonically divided by the conic.

Construction: The fifteen i-lines through a given vertex of the hexagon are determined by joining the vertex to the fifteen Pascal points determined by the remaining five vertices.

The sixteen k-lines through a given Pascal point are determined by joining the given point to the sixteen Pascal points determined by using three of the vertices of the hexagon used in the given point and one vertex not used in the given point. Thus one k-line through 36,45 would pass through 34,62.

NON-ANALYTIC FUNCTIONS OF A COMPLEX VARIABLE

By LOUIS BRAND, University of Cincinnati

1. From the theory of plane quaternions $a+bi+cj$ two types of plane vector analysis have arisen. The one interprets $a+bi$ as a vector w in the complex plane and carries on its operations according to the algebra of quaternions. The other interprets $bi+cj$ as a real vector w and decomposes the quaternion product w_1w_2 into its scalar and vector parts, which are thereafter used separately as "products."

To indicate the interpretation used, we shall denote complex and real vectors by italic and bold-face letters respectively. Thus the plane vector whose components are u, v may be written as

$$w = u + vi \text{ or } \boldsymbol{w} = u\boldsymbol{i} + v\boldsymbol{j}.$$

In the first case

$$w_1w_2 = (u_1u_2 - v_1v_2) + (u_1v_2 + u_2v_1)i; \tag{1}$$

in the second

$$\boldsymbol{w}_1\boldsymbol{w}_2 = -(u_1u_2 + v_1v_2) + (u_1v_2 - u_2v_1)\boldsymbol{k}. \tag{2}$$

In Gibbs' notation

$$u_1u_2 + v_1v_2 = \boldsymbol{w}_1 \cdot \boldsymbol{w}_2, \quad u_1v_2 - u_2v_1 = \boldsymbol{k} \cdot \boldsymbol{w}_1 \times \boldsymbol{w}_2.$$

Let $w' = u - vi$ denote the conjugate of w; then from (1) and (2) we find that

$$w_1' w_2 = \boldsymbol{w}_1 \cdot \boldsymbol{w}_2 + i\boldsymbol{k} \cdot \boldsymbol{w}_1 \times \boldsymbol{w}_2, \quad w_1w_2' = \boldsymbol{w}_1 \cdot \boldsymbol{w}_2 - i\boldsymbol{k} \cdot \boldsymbol{w}_1 \times \boldsymbol{w}_2,$$

so that

$$\boldsymbol{w}_1 \cdot \boldsymbol{w}_2 = \tfrac{1}{2}(w_1w_2' + w_1' w_2), \tag{3}$$

$$\boldsymbol{k} \cdot \boldsymbol{w}_1 \times \boldsymbol{w}_2 = \tfrac{1}{2}i(w_1w_2' - w_1' w_2). \tag{4}$$

From (3) and (4) we have the conditions

$$w_1 w_2' + w_1' w_2 = 0, \quad w_1 w_2' - w_1' w_2 = 0$$

for the perpendicularity and parallelism respectively of the complex vectors w_1, w_2.

We next introduce the operators

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} \text{ and } \nabla = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$$

in the real and complex planes, and their conjugates ∇'. Then if $\phi(x, y)$ is a real scalar point function,

$$\text{grad } \phi = \nabla\phi = \frac{\partial \phi}{\partial x} + i\frac{\partial \phi}{\partial y};$$

and in view of (3) and (4),

$$\text{div } \boldsymbol{w} = \nabla \cdot \boldsymbol{w} = \tfrac{1}{2}(\nabla w' + \nabla' w), \tag{5}$$

$$\boldsymbol{k} \cdot \text{rot } \boldsymbol{w} = \boldsymbol{k} \cdot \nabla \times \boldsymbol{w} = \tfrac{1}{2}(\nabla w' - \nabla' w) i, \tag{6}$$

$$\nabla^2 \phi = \nabla \cdot \nabla \phi = \nabla \nabla' \phi. \tag{7}$$

It is often advantageous to introduce the variables

$$z = x + iy, \quad z' = x - iy$$

instead of x, y in the complex plane. Thus to any function $w(x, y)$ there corresponds uniquely a function $w(z, z')$. Then we find that

$$\frac{\partial w(z, z')}{\partial z} = \tfrac{1}{2}\nabla' w(x, y), \quad \frac{\partial w(z, z')}{\partial z'} = \tfrac{1}{2}\nabla w(x, y),$$

whence the operational identities

$$\frac{\partial}{\partial z} = \tfrac{1}{2}\nabla', \quad \frac{\partial}{\partial z'} = \tfrac{1}{2}\nabla. \tag{8}$$

Therefore on changing variables from x, y to z, z' in (5), (6), (7), we obtain:

$$\text{div } \boldsymbol{w} = \frac{\partial w'}{\partial z'} + \frac{\partial w}{\partial z} \tag{5}'$$

$$\boldsymbol{k} \cdot \text{rot } \boldsymbol{w} = \left(\frac{\partial w'}{\partial z'} - \frac{\partial w}{\partial z}\right) i \tag{6}'$$

$$\nabla^2 \phi = 4\frac{\partial^2 \phi}{\partial z \partial z'}. \tag{7}'$$

We are now in possession of a dictionary that will paraphrase the theorems for

real vector functions $\boldsymbol{w}$ into the corresponding theorems for non-analytic functions w in the complex plane.

2. We first seek to paraphrase the theorem that the derivative of $\boldsymbol{w}$ in the direction of the unit vector $\boldsymbol{t}$ is

$$\frac{d\boldsymbol{w}}{ds} = \boldsymbol{t} \cdot \nabla \boldsymbol{w}, \tag{9}$$

where we assume the existence and continuity of the partial derivatives involved. Let $\boldsymbol{t} = e^{i\theta}$; then from (3) we have

$$\frac{dw}{ds} = \tfrac{1}{2} e^{i\theta} \nabla' w + \tfrac{1}{2} e^{-i\theta} \nabla w.$$

Since $dz = e^{i\theta} ds$ in the direction $\boldsymbol{t}$,

$$\frac{dw}{dz} = \tfrac{1}{2} \nabla w e^{-2i\theta} + \tfrac{1}{2} \nabla' w. \tag{10}$$

Thus dw/dz will be independent of θ when and only when $\nabla w = 0$, a condition that is precisely a synthesis of the Cauchy-Riemann equations. In view of (8), (10) may also be written in the simple form[1]

$$\frac{dw}{dz} = \frac{\partial w}{\partial z'} e^{-2i\theta} + \frac{\partial w}{\partial z}. \tag{11}$$

From (11) we see that if the vector dw/dz is drawn from a point, then as θ varies from 0 to 2π its end-point travels twice in the opposite direction about a circle of center w_z and of radius $|w_{z'}|$—the *Kasner circle*.[2] Moreover if $w_z w_{z'} \neq 0$ and

$$w_z = ae^{i\alpha}, \quad w_{z'} = be^{i\beta},$$

$$\frac{dw}{dz} = be^{i(\beta - 2\theta)} + ae^{i\alpha};$$

hence $|dw/dz|$ attains its maximum value $a+b$ when $\theta = \frac{1}{2}(\beta - \alpha)$ and its minimum value $|a-b|$ when $\theta = \frac{1}{2}(\beta - \alpha) + \frac{1}{2}\pi$. These are the principal directions of dw/dz at the point z.[3]

3. We next consider certain vector fields w of particular importance in mathematical physics.

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[1] G. Calugaréano, *Sur les fonctions polygènes d'une variable complexe*, Comptes rendus, vol. 186 (1928), p. 930.

[2] E. Kasner, *General theory of polygenic functions*, Proc. Nat. Acad. of Sciences, vol. 14 (1928), p. 75.

[3] Tissot, *Sur les cartes géographiques*, Comptes rendus, vol. 49 (1859), p. 673. Hedrick, Ingold, and Westfall, *Theory of non-analytic functions of a complex variable*, Journal de mathématiques, Ser. 9, vol. 2 (1923), p. 335.

The Irrotational Field: rot $\boldsymbol{w}=0$. Then $\boldsymbol{w}$ is expressible as the gradient of a scalar point function:

$$\boldsymbol{w} = \nabla\phi \text{ where } \phi = \int_{r_0}^{r} \boldsymbol{w}\cdot d\boldsymbol{r}.$$

In the complex plane, therefore, $w(z, z')$ is irrotational when and only when

$$\frac{\partial w}{\partial z} - \frac{\partial w'}{\partial z'} = 0;$$

then there exists a real function $\phi(z, z')$ such that

$$w = \frac{\partial \phi}{\partial z'}, \quad \text{and} \quad \phi = \int_{z_0}^{z} (w'dz + wdz').$$

The field lines are the orthogonal trajectories of the curves $\phi=\text{const.}$

The Sourceless Field: div $\boldsymbol{w}=0$. Then $\boldsymbol{w}$ is expressible as

$$\boldsymbol{w} = \boldsymbol{k}\times\nabla\phi \text{ where } \phi = \int_{r_0}^{r} \boldsymbol{w}\times\boldsymbol{k}\cdot d\boldsymbol{r}.$$

In the complex plane, therefore, $w(z, z')$ is sourceless when and only when

$$\frac{\partial w}{\partial z} + \frac{\partial w'}{\partial z'} = 0;$$

then there exists a real function $\phi(z, z')$ such that

$$w = i\frac{\partial \phi}{\partial z'} \quad \text{and} \quad \phi = i\int_{z_0}^{z} (w'dz - wdz').$$

The field lines are the curves $\phi=\text{const.}$

We summarize these results in the

THEOREM. *If $\phi(z, z')$ is a real function with continuous partial derivatives, then $\phi_{z'}$ and $i\phi_{z'}$ give respectively an irrotational field with lines orthogonal to $\phi=const.$, and a sourceless field with the lines $\phi=const.$*

The above results give a simple method for decomposing a plane vector function into an irrotational and a sourceless part. For if

$$\boldsymbol{w} = \boldsymbol{w}_1 + \boldsymbol{w}_2, \quad \text{rot}\,\boldsymbol{w}_1 = 0, \quad \text{div}\,\boldsymbol{w}_2 = 0,$$

there exist real functions ϕ, ψ such that

$$w = \frac{\partial \phi}{\partial z'} + i\frac{\partial \psi}{\partial z'} = \frac{\partial}{\partial z'}(\phi + i\psi).$$

Hence the vector

$$\phi + i\psi = \int w(z, z')dz' \quad (z \text{ constant})$$

is determined to an arbitrary additive $f(z)$; the decomposition is therefore unique.

If the vector field $w(z, z')$ is both irrotational and sourceless, rot $\boldsymbol{w} = 0$ implies that

$$w = \frac{\partial \phi}{\partial z'}; \text{ then } w' = \frac{\partial \phi}{\partial z}, \quad \operatorname{div} \boldsymbol{w} = 2\frac{\partial^2 \phi}{\partial z \partial z'} = 0.$$

Hence ϕ is a real harmonic function, w is an analytic function of z', and w' is an analytic function of z.

THEOREM. *If w is an analytic function of z', the field is both irrotational and sourceless. The conjugate of an irrotational and sourceless field vector is an analytic function of z.*

4. Finally let us paraphrase the integral theorems in the plane:

$$\int \operatorname{div} \boldsymbol{w} d\sigma = \oint \boldsymbol{n} \cdot \boldsymbol{w} ds = \oint \boldsymbol{k} \cdot \boldsymbol{w} \times d\boldsymbol{r}, \tag{12}$$

$$\int \boldsymbol{k} \cdot \operatorname{rot} \boldsymbol{w} d\sigma = \oint \boldsymbol{w} \cdot d\boldsymbol{r}, \tag{13}$$

which hold when $\boldsymbol{w}$ and $\nabla \boldsymbol{w}$ are continuous. Here $\boldsymbol{n}$ is a unit external normal to the closed curve bounding the area over which the left-hand integrals are taken; and $\boldsymbol{k}$ is a unit vector normal to the plane in the sense of progression of a right-handed screw turned in the direction of the circuit integrals on the right.

Using (3), (4), (5), (6), the paraphrases of (12) and (13) are

$$\int (\nabla w' + \nabla' w) d\sigma = i \oint (w dz' - w' dz), \tag{14}$$

$$i \int (\nabla w' - \nabla' w) d\sigma = \oint (w dz' + w' dz). \tag{15}$$

Multiply (14) by i and add to (15); then

$$i \int \nabla w' d\sigma = \oint w' dz,$$

and on replacing w' by w, we have

$$\oint w dz = i \int \nabla w d\sigma = 2i \int \frac{\partial w}{dz'} d\sigma. \tag{16}$$

When w is analytic over the area, $\oint w dz = 0$; thus (16) is a generalization of

Cauchy's Integral Theorem. With $w=z'$ in (16) we obtain the usual circuit integral for a plane area; and with $w=zz'$ we obtain expressions for the first moment of a plane area as a circuit integral.

From (16) we may deduce the generalization of Cauchy's integral formula,

$$f(\zeta) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-\zeta}\,dz - \frac{1}{\pi}\int \frac{\partial \bar{f}/\partial z'}{z-\zeta}\,d\sigma,$$

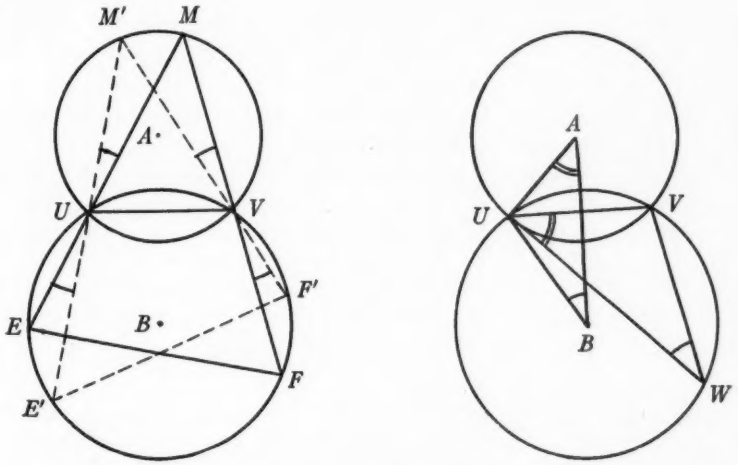
where $f(z) \equiv \phi(x, y) \equiv \bar{f}(z, z')$.

ON TWO INTERSECTING SPHERES

By N. A. COURT, University of Oklahoma

I. Introduction. The object of this paper is to extend to two intersecting spheres some properties of two intersecting circles.[1] For the convenience of the reader I shall reproduce here those properties of the circles which will be referred to in what follows.

(a) *The two lines joining the two (real) points of intersection of two circles to a variable point of one of these circles determine in the other circle a chord of constant length.*



Let U, V, be the points common to the two circles (A), (B) (Fig. 1), and let E, F, be the traces on (B) of the lines MU, MV, joining U, V, to any point M of (A). If M' is any other position of M, and E', F', the corresponding pair of points on (B), it is readily seen that the arcs EE', FF', are equal, and that EF, $E'F'$, are the diagonals of an isosceles trapezoid; hence the proposition.

[1] Nathan Altshiller-Court. *Sur deux cercles secants.* Mathesis, vol. 39 (1925), p. 453.

(b) When the point M coincides with U (Fig. 2), the point F also coincides with U, the line MUE becomes the tangent to (A) at U, and the chord EF is the segment UW intercepted on this tangent by the circle (B).

(c) Let A, B, denote the centers of (A), (B), (Fig. 2). The two triangles AUB, UVW, are equiangular, hence $UW:AB = UV:AU$; or, if $AU=a$, $AB=d$, $UV=c$, and $UW=y$, then $ya=cd$. This relation gives the length of the constant chord $EF=y$ of the above theorem (a).

II. THEOREM. *The variable cone having for base the (real) circle of intersection of two spheres and for vertex a variable point of one of these spheres cuts the other sphere again along a circle whose radius is constant.*

The cone Σ having for base the common (real) circle (S) of the two given spheres (A), (B), and for vertex any point M of (A) penetrates into (B) along (S), hence Σ emerges from (B) along another circle (T).

The principal plane of Σ is determined by the line MS joining M to the center S of (S) and by the perpendicular at S to the plane of (S); hence the principal plane of Σ contains the centers A, B, of (A), (B), and cuts these spheres along great circles. This principal plane cuts (S) along the principal diameter UV, and the principal elements MU, MV, of Σ meet (B) again in the ends E, F, of a diameter of (T).[1] The proposition thus follows immediately from the case of the plane (see Ia).

III. If $AU=a$, $AB=d$, $UV=c$, and $EF=y$, we have (see Ic)

$$ya = cd. \tag{1}$$

We have thus a simple expression for the length y of the diameter EF of (T) in terms of the fixed elements of the two given spheres.

When the vertex M of the cone Σ coincides with a point L of the circle (S), the cone degenerates, and (T) becomes the circle of intersection of the sphere (B) with the tangent plane to (A) at L; the formula (1) gives the diameter y of this circle.

The segment BT joining B to the center T of the circle (T) is equal to the distance h of B from the plane of (T). If we denote by b the radius of (B) we obtain from the right triangle BTE and the formula (1),

$$h^2 = (4a^2b^2 - d^2c^2):4a^2. \tag{2}$$

If the sphere (A) is bisected by the sphere (B), then $c=2a$, and $d^2=b^2-a^2$, and the formulas (1), (2), give

$$y = 2d, \quad h = a.$$

The reader may formulate verbal statements expressing these results.

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[1] Rouché et Comberousse. *Traité de géométrie*, vol. II, pp. 223–224. Gauthier-Villars. Paris, 1900.

IV. Consider the cone Σ' having (S) for its base and its vertex on the sphere (B). The formula (1) gives for the diameter x of the circle of intersection of Σ' with (A),

$$xb = cd;$$

hence

$$cd = yb = xa, \text{ or } x:y = a:b.$$

Thus: *The two cones having for base the common circle of two spheres and for vertices any two points on the two spheres cut these spheres again along two circles whose radii are proportional to the radii of the spheres on which the circles are situated.*

As a limiting case we have: *The two tangent planes to two spheres at a point common to these spheres cut the spheres along two circles whose radii are proportional to the radii of the spheres on which the circles are situated.*

V. If the sphere (A) and the circle (S) remain fixed while the sphere (B) varies describing the coaxal pencil having (S) for basic circle, we have, by (1),

$$y:d = c:a,$$

hence: *The cone passing through the basic circle of a coaxal pencil of spheres and having its vertex on a given sphere of the pencil cuts the spheres of the pencil again along circles whose radii are proportional to the distances of the centers of the respective spheres from the center of the given sphere.*

VI. When the vertex M of the cone Σ lies on (S) the preceding proposition takes the following form: *The tangent plane to a sphere of a coaxal pencil at a point of the basic circle of the pencil cuts the other spheres along circles whose radii are proportional to the distances of the centers of the respective spheres from the center of the first sphere.*

The special case of the last two propositions when (S) is a great circle on (A) is left for the consideration of the reader.

VII. *If p, q, are the squares of the radii of two spheres, t the square of their line of centers, and s the square of the radius of their common circle, we have*

$$4st = 4pq - (p + q - t)^2. \tag{3}$$

If u is the power, with respect to either of the two given spheres (P), (Q), of the trace X of the radical plane of (P), (Q), on their line of centers PQ, we have

$$u = XP^2 - p = XQ^2 - q,$$

and hence

$$XP^2 + XQ^2 = 2u + p + q. \tag{4}$$

On the other hand,

$$XP^2 - XQ^2 = p - q,$$

and we have, both in magnitude and in sign,

$$(XP + XQ)(XP - XQ) = p - q,$$

or

$$(XP + XP + PQ)(QX + XP) = p - q.$$

Hence we have, both in magnitude and in sign,

$$2PX \cdot PQ = PQ^2 + (p - q);$$

and, similarly,

$$2QX \cdot QP = QP^2 + (q - p).$$

Multiplying these two equalities and putting $PQ^2 = t$, we have, both in magnitude and in sign,

$$4PX \cdot XQ \cdot t = t^2 - (p - q)^2. \tag{5}$$

Multiplying (4) by $2t$ and adding to (5) we have, after simplification,

$$4ut = (p + q - t)^2 - 4pq.$$

Now u and s are equal in magnitude and opposite in sign, hence the announced relation (3).

In the above proof no use has been made of points on either sphere, hence the formula (3) is valid for any two spheres whose centers are real and the squares of whose radii, as well as the square of the radius of their common circle, are either positive, or negative.

If the spheres (P), (Q), are orthogonal, we have $t = p + q$, and (3) becomes $4(p+q)s = 4pq$ or[1]

$$\frac{1}{s} = \frac{1}{p} + \frac{1}{q}.$$

VIII. The formula (3) applied to the spheres (A), (B), (see II, III) becomes

$$c^2d^2 = 4a^2b^2 - (a^2 + b^2 - d^2)^2; \tag{6}$$

hence, using the formula (1),

$$y^2a^2 = 4a^2b^2 - (a^2 + b^2 - d^2)^2. \tag{7}$$

We have thus the value of the diameter $EF = y$ in terms of the radii of the given spheres and of the length of their line of centers.

From the formulas (6), (2), we have

$$h = (a^2 + b^2 - d^2) : 2a. \tag{8}$$

[1] Nathan Altshiller-Court. *On five mutually orthogonal spheres.* Annals of Mathematics, vol. 30 (1929), p. 614.

IX. If (A), (B), are orthogonal, we have $a^2+b^2=d^2$, and (7) gives $y=2b$; hence:

The cone passing through the common circle of two orthogonal spheres and having its vertex on one of these spheres cuts the second sphere along a great circle.

This result also follows from (8), since h in this case becomes zero.

The same formula (8) shows that the tangent plane to one of two orthogonal spheres at a point common to the two spheres passes through the center of the second sphere, as was to be expected.

X. As the vertex M of Σ varies on (A) the plane of the circle (T) remains at a fixed distance from the center B of (B), hence the plane of (T) envelopes a sphere (C) concentric with (B). The sphere (C) is tangent to the planes which touch (A) at the points along (S) (see III), i.e., (C) is inscribed in the cone which circumscribes (A) along (S). Stated in this form the proposition admits of the following projective generalization:

Given two quadric surfaces (A), (B), intersecting along two conics (S), (S'), the cone projecting one of these conics, say (S), from a variable point of one of the surfaces, say (A), cuts the second surface (B) along a conic whose plane envelopes a third quadric (C) tangent to (B) along the conic (S'). Moreover (C) is inscribed in the cone circumscribing (A) along the conic (S).

THE DEVELOPMENT OF THE FUNDAMENTAL CONCEPTS OF INFINITESIMAL ANALYSIS

By F. L. WREN, George Peabody College for Teachers, and
J. A. GARRETT, A. & M. College, Monticello, Arkansas

The mathematical interests of the ancient Greeks were primarily centered in geometry and geometrical constructions. As early as the fifth century B.C. they found themselves confronted with three problems which they soon realized could not be solved by the use of the unmarked straightedge and compasses alone.[1] These problems were the trisection of an arbitrary angle, the quadrature of the circle, and the duplication of the cube (or Delian problem).

The problem of squaring the circle has, throughout the ages, attracted interested attention. Among the first to attempt a solution was Antiphon (c. 430 B.C.), whose method was that of inscribing a regular polygon and exhausting the area by doubling the number of sides. By applying this method to both inscribed and circumscribed polygons, Archimedes (c. 225 B.C.) was able to calculate that the value of π lies between 3 1/7 and 3 10/71. Among other methods devised for squaring the circle were those using the quadratrix of Hippias (c. 425 B.C.), and the spiral of Archimedes.

[1] Sir Thomas Heath, *A History of Greek Mathematics*, Clarendon Press, Oxford, 1921, Vol. I, p. 219.

The problem of trisecting an angle "presumably arose from attempts to continue the construction of regular polygons after that of the pentagon had been discovered."[1] Out of the efforts to arrive at this construction the development and use of the conchoid by Nicomedes (c. 180 B.C.) and the use of conics by Pappus (c. 300 A.D.) are probably the most interesting.

"Eratosthenes (c. 230 B.C.) in his letter to Ptolemy III, relates that one of the tragic poets introduced Minos on the stage erecting a tomb for his son, Glaucus; and then, deeming the structure too mean for a royal tomb, he said 'Double it, but preserve the cubical form' . . . and continues: 'Later (in the time of Plato), so the story goes, the Delians, who were suffering from a pestilence, being ordered by the oracle to double one of their altars, were thus placed in the same difficulty.'"[2] Out of the research on the Delian problem we have the method of Archytas (c. 400 B.C.) in which he employed a right cone, a cylinder, and a tore of zero inner diameter; the probable discovery and use of conic sections by Menaechmus (c. 350 B.C.);[3] the use of the conchoid by Nicomedes and the invention and use of the cissoid by Diocles (c. 180 B.C.).[4]

Thus we see that out of the research on these three famous geometric problems of antiquity there developed many new curves whose properties were unknown. The interest in the properties of these curves, and curves of a similar nature, has been one of the strongest of mental stimuli in the development of the methods of infinitesimal analysis.

While Antiphon had made direct use of the method of exhaustion in his efforts at the quadrature of the circle, the discovery of this method is attributed by some to Hippocrates of Chios (c. 460 B.C.) and by others to Eudoxus (c. 370 B.C.). In the mathematical researches attributed to these men are found the first attempts to employ a technique that was in any way based upon a concept of infinitesimals or the application of a limiting process. The method of exhaustion may thus be considered as a form of infinitesimal analysis whose fundamental hypotheses are contained in the two propositions:

"(1) If A and B be two magnitudes of the same kind, of which A is greater, and there be taken from A more than its half (or any other fraction) and from the remainder more than its half (or any other fraction) and so on, the ultimate remainder will be less than B."

"(2) Let there be two magnitudes P and Q, both of the same kind and let a succession of magnitudes X_1, X_2, X_3, etc., be each nearer and nearer to P, so that any one X_r, shall differ from P less than half as much as its predecessor differed. Let Y_1, Y_2, Y_3, etc., be a succession of magnitudes similarly related to

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[1] Heath, op. cit., p. 235.

[2] G. J. Allman, *Greek Geometry from Thales to Euclid*, Longmans, Green & Co., 1889, p. 85.

[3] Heath, op. cit., pp. 251–255.

[4] D. E. Smith, *History of Mathematics*, Ginn and Co., 1923, Vol. I, p. 118.

Q, and let the ratios $X_1 : Y_1$, $X_2 : Y_2$, etc., be all the same with each other and the same as $A : B$. Then $P : Q :: A : B$."[1]

Archimedes, probably the greatest mathematician of all antiquity, and one of the greatest in the history of the world, was an original investigator in problems relating to the quadrature of curvilinear plane figures and the quadrature and cubature of curved surfaces. "These investigations, . . . , actually (in the words of Chasles) 'gave birth to the calculus of the infinite conceived and brought to perfection successively by Kepler, Cavalieri, Fermat, Leibniz, and Newton.' He performed in fact what is equivalent to *integration* in finding the area of a parabolic segment, and of a spiral; the surface and volume of a sphere and a segment of a sphere, and the volumes of any segments of the solids of revolution of second degree."[2] He first discovered the solution of the quadrature of the parabola through application of principles of mechanics but later confirmed his result by applying the method of exhaustion. This latter method consisted in exhausting the area of a segment of a parabola by means of a series of triangles inscribed "in the recognized manner," i.e., the vertex of the triangle inscribed in a segment being the point of contact of the tangent drawn parallel to the base of the segment. By this process he was able to express the value of the parabolic area in the form $A(1+\frac{1}{4}+(\frac{1}{4})^2+(\frac{1}{4})^3+\cdots)$ where A was the area of the original inscribed triangle. Having obtained this series Archimedes, instead of passing to the limit, used the method of *reductio ad absurdum* to determine its value.

Although Archimedes made several outstanding contributions to Stereometry, particularly his work on *Spheroids and Conoids*, interest in this field of endeavor did not remain alive. This lack of interest was due, in part, to the influence of the Athenian school, in part, to Apollonius (c. 225 B.C.) whose work on conics drew attention in that direction, but, for the most part, it was due to the inflexibility and lack of generalization of the method of exhaustion. It remained for the imagination of later races to conceive of other methods by the aid of which further advancement might be possible.

The years of Roman domination of the civilized world and the subsequent Dark Ages were not notably fruitful in mathematical advancement since the Romans were not given to abstract thinking and did not encourage it. The political and social condition of the western world did well to keep the germ of knowledge alive, it certainly did not furnish a healthy field for its development and growth. The more or less complacent acquiescence to the will of the Church was also a factor in preventing progress in the fields of mathematics and the sciences. It was not until the Renaissance that interest in mathematical research of any significance began to be revived.

In Germany occurred the first really successful revolt from the ecclesiastical domination that had prevented fruitful scientific development. Among the

[1] J. Gow, *A Short History of Greek Mathematics*, Cambridge University Press, 1884, pp. 171–172.

[2] Heath, op. cit., Vol. II, p. 20.

leaders in this movement we find Johannes Kepler (1571–1630) who was the first to introduce into geometry the concepts of infinitely small and infinitely great quantities. He proposed the atomistic theory of geometric figures, which states that they can be considered as composed of an infinite number of indivisible units; thus, lines are aggregates of points, surfaces of lines, and solids of surfaces. Extending this idea further, a sphere may be considered as the sum of cones which are in their final form coincident with the radii of the sphere. Likewise, a cone may be considered to be the sum of triangles formed by passing planes through the vertex of the cone. These schemes bear a very close resemblance to the methods used by Archimedes and also to the method of summation later used in the integral calculus.

A great deal of Kepler's original research was incited by an interest in the work of Archimedes on solids. In his *Spheroids and Conoids*, Archimedes had created many solids whose volumes he had determined; these solids were conceived as generated by the revolution of conics, spirals, and circular segments about their respective axes. Kepler published in his *Nova Stereometria* an extension of this work on solids. This extension was accomplished by means of considering the solids generated by the revolution of these curves about a chord, a tangent, or even an extraneous line. In this way he added ninety new solids to those proposed by Archimedes, the simplest of which he solved. The more complex ones, however, remained as a test of the ingenuity of later mathematicians. The second chapter of this book dealt with the question of the proper method of gauging the contents of a wine barrel, and this lead him into a discussion on cross-sectional areas in which we find the foundation of the rule on maxima and minima.[1] By estimating the circumference of an ellipse whose axes are $2a$ and $2b$ to be $\pi(a+b)$, he anticipated the problem of rectification of curves; another problem also anticipated by Kepler was that of inverse tangents.[2]

Cavalieri (1598–1647) was lead by Kepler's method of integration into the development of his method of indivisibles. "In order to prove his fitness for the Chair of Mathematics at the University of Bologna he submitted, in 1629, the manuscript of his famous work, *Geometria Indivisibilibus Continuorum Nova quadam ratione promota*, which he published in 1635."[3] He considered geometrical magnitudes to be composed of indivisible elements which could be formed by continually slicing with parallels, and then used the ratio of the sum of these indivisible elements to a known constant. A good example of this method may be seen in the relation between the sum of the indivisible elements of a triangle as compared to the area of a parallelogram with the same base and altitude. Cavalieri considered the lines that fill a triangle to be in arithmetic progression

[1] J. E. Montucla, *Histoire des Mathématiques*, Henri Agone, Paris, 1802, Tome 2, p. 31.

[2] M. Cantor, *Vorlesungen über Geschichte der Mathematik*, B. G. Teubner, Leipzig, 1913, Band 2, p. 829.

[3] D. E. Smith, *Source Book in Mathematics*, McGraw-Hill Book Co., 1929, p. 605.

beginning with the apex 1 and going to the base n. The sum of these numbers is $n(n+1)/2$, but each line of the parallelogram is n and consequently its area is n^2, and the ratio of these two quantities is $\frac{1}{2}$ since n is infinite.

Although Cavalieri, by means of his method of indivisibles, was able to solve most of the solids that Kepler had proposed as well as the theorem of Pappus with respect to a volume generated by the revolution of a plane figure about an axis, it was criticized as being unscientific. If a line has no width, how could any number of them be found so as to fill a plane figure? If a plane has no thickness how could it be possible that even an infinite number might form a solid? Whether scientific or not the method exerted an enormous influence on the period that followed and served as a method of integration for forty or fifty years.

The claim of Cavalieri to the method of indivisibles was contested by friends of Roberval (1602–1675). Mersenne (1588–1648) had, in 1629, proposed the quadrature of the cycloid and, in 1634, Roberval sent him a solution of this problem in which he employed the method of indivisibles. Walker[1] lends support to Mersenne's statement "that Roberval's method of indivisibles was worked out between 1628 and 1634." She further points out, however, that there seemed to be no controversy between the two mathematicians themselves.[2] According to Chasles, Roberval was much more rigorous in the application of the method than was Cavalieri.[3]

Although not contributing directly to the methods of infinitesimal analysis the influence of Descartes (1596–1650) is by no means to be overlooked nor minimized. Descartes's greatest discovery was his coördinate geometry. This did not consist merely of applying algebra to geometry, for others, beginning with Archimedes, had used algebra in connection with geometry. Descartes systematized the method, introduced the notion of variables and constants into geometry, conceived of curves as generated by a moving point, referred these curves to two lines perpendicular to each other and represented them by equations involving two variables, the relation of these variables being determined by the distances from the two lines of reference. It was this notion of expressing curves by algebraic equations that made possible the step from geometry to analysis and thus paved the way for calculus.

In his *Arithmetica Infinitorum*, published in 1655, Wallis (1616–1704) gave a treatment of quadratures that summed up the ideas of integral calculus, without the mechanism, as we have it today. In this treatise, he assumed the principle of continuity first advanced by Kepler and extended the idea to include the analytic concepts advanced by Descartes. Like other mathematicians of his

[1] Evelyn Walker, *A Study of the Traité des Indivisibles of Gilles Persone de Roberval*, Teachers College Columbia University, Contribution to Education #446, Bureau of Publications, Teachers College, N. Y., 1932, p. 143.

[2] Evelyn Walker, op. cit., p. 15.

[3] M. Chasles, *Aperçu Historique sur l'Origine et le Développement des Méthodes en Géométrie*, M. Hayez, Bruxelles, 1837, p. 61.

time, he considered geometric plane figures "as if" composed of an infinite number of lines, but was careful to say, that really these lines were parallelograms of infinitesimal width.

Pascal's (1623–1662) contribution to the theory of integration was embodied in a series of letters on certain problems concerning the properties of the cycloid. He employed practically every artifice known at that time, in particular, may be noted his summation of sines, the evaluation of "ungulae"[1] which had been suggested by St. Vincent (1584–1667), and the triangular and pyramidal sums. He called triangular sums "indivisibles" with respect to simple sums and pyramidal sums "indivisibles" with respect to triangular sums.[2] Pascal employed the method of Cavalieri with eminent success. His researches, according to d'Alembert, closely approach the integral calculus, and form the connection between the methods of Archimedes and Newton.[3]

Up to this point the narrative has been confined to the introduction to the integral calculus. Along with this evolution of the theory of integration, there developed out of problems concerning maxima and minima some of the concepts that finally led to the differential calculus. The problem of tangents was an inheritance from the ancients. In the early part of the seventeenth century interest in the problem was revived. Three different view points of this period were:

(1) tangents are secants whose two points of intersection with the curve coincide;

(2) tangents are prolongations of the sides of an inscribed polygon of an infinite number of sides;

(3) tangents are the direction of a curve at any particular moment.

The third view point was that of Roberval who recognized the value of his idea and at the same time felt its limitations. He succeeded in applying it to only a few curves; that he failed to make a general application was due to his inability to devise a scheme that was not dependent on a special property of the curve in question. The method bears a remarkable analogy to Newton's method of fluxions.

Fermat's (1601–1665) method was dependent on the principle of maxima and minima, first enunciated by Kepler, that in the neighborhood of a maximum value of a variable its increment becomes evanescent. For the determination of a maximum, he employed a rule which may be stated as follows: To determine any maximum value of a function $f(x)$, substitute $x+e$ for x, and, from the resulting expression, subtract $f(x)$, divide by e and eliminate the remaining e's. It is readily seen that this method is essentially that of the differential calculus. For this reason Lagrange attributed the discovery of the calculus to

[1] Ungulae denoted hoof-shaped solids, such as the frusta of cylinders or cones cut off by planes that are not parallel to one another.

[2] M. Marie, *Histoire des Sciences Mathématiques et Physiques*, Gauthiers-Villars, Paris, Tome IV, 1884, p. 193.

[3] W. W. R. Ball, *A Short History of Mathematics*, The Macmillan Co., 1893, p. 288.

Fermat. M. Poisson challenged this in the following words: "But the calculus consists of an ensemble of rules for finding immediately the differential of all functions, rather than the use that can be made of these infinitely small variations to solve such or like species of problems; and under this definition the creation of the differential calculus does not go back of Leibniz, author of the algorithm and of the notation which has so generally prevailed since the origin of calculus and to which infinitesimal analysis is principally responsible for its progress."[1]

Fermat also solved various problems of integration employing an ingenious application of arithmetic and geometric means. His demonstrations were rigorous but, even as in his method of tangents, they lacked a generality sufficient to affect the general trend of the progress of the science.

Isaac Barrow (1630–1677) extended and systematized the methods used by Fermat by introducing along with the e, another infinitesimal, a. He made use of the a and e somewhat as Leibniz used the dy and dx. Child is of the opinion that Barrow's method was not an extension of Fermat's, that the two methods were discovered independently, and that it was a mere coincidence that both men used the same letter e as the increment of x. He is also quite positive in giving Barrow the credit for the original concept of the characteristic triangle which Leibniz used so effectively at a later date.[2] All of Barrow's work was geometrical, his rule being devised primarily for the purpose of constructing tangents. The rule, however, is more important than that for which it was created, since by using it, he was able to perform genuine differentiations and integrations, though in a geometric form. In fact, Barrow created a calculus, similar in principle to the calculus of Leibniz and Newton, but applicable only to geometry.

Sir Isaac Newton (1642–1727) was a pupil of Barrow's and was greatly influenced by his mathematical efforts and those of Wallis. His first essay was the result of a study of Wallis's *Arithmetica Infinitorum*. Wallis had been unable to expand binomials with fractional exponents and consequently had been unable to apply his methods to the area of a circle, since this involved the expansion of the expression $y=(1-x^2)^{1/2}$. Newton, by a scheme of interpolation, had found the required series and effected the quadrature. Later he discarded interpolation and discovered the binomial theorem for all real exponents, by the aid of which he found expressions in the form of infinite series for areas and arcs of curves.[3]

Newton termed his plan of interpolation and the binomial theorem as his first and second methods. His third method, which was a new and independent system of infinitesimal analysis, he called the method of fluxions. It seems that

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[1] Chasles, op. cit., p. 63.

[2] J. M. Child, *The Geometrical Lectures of Isaac Barrow*, Open Court Publishing Co., 1916, pp. 13–14.

[3] Ball, op. cit., p. 328.

he had given serious thought to this method in 1665 and by the following year had formulated it into a rather definite system, but that he did not communicate with any of his friends about it until 1669 when he released the secret to his friend and former teacher, Isaac Barrow. In 1676, in response to a request from Leibniz for information on the inverse problem of tangents, he wrote that he was in possession of a general method of solution, but, in accordance with the spirit of the times, concealed it in a cipher, which when deciphered contained not only a statement of the problem but also a hint as to its solution. The statement bore the following meaning "given any equation whatsoever, involving fluent quantities, to find the fluxions, and vice versa."[1]

The term, fluxion, indicates motion. The idea of the fluxional calculus developed from the concept that a geometrical magnitude was the result of continuous motion of a point, line, or plane. This motion, speaking of plane curves, could be considered, when referred to coördinate axes, as the resultant of two motions, one in the direction of the X-axis and the other in the direction of the Y-axis. The velocity of the X-component and the Y-component were called "fluxions" by Newton. He represented them by $\dot{x}$ and $\dot{y}$. The velocity of a point is represented by an equation involving the fluxions $\dot{x}$ and $\dot{y}$; reciprocally, the arc is the "fluent" of the velocity of a moving point, so the x and y values are the fluents of the fluxions $\dot{x}$ and $\dot{y}$. The problem of fluxions is, thus, dual in nature:

"1. The length of the space described being continually given; to find the velocity of the motion at any time proposed.

"2. The velocity of the motion being continually given; to find the length of the space described at any time proposed."[2]

Fluxions were not considered as infinitesimals, but "moment of a fluxion" was the term introduced to denote an "indefinitely small" quantity. "If the moments of x be represented by the product of its celerity $\dot{x}$ into an indefinitely small quantity (that is by $\dot{x}o$), the moment of y will be $\dot{y}o$, since $\dot{x}o$ and $\dot{y}o$ are to each other as $\dot{x}$ and $\dot{y}$."[3] These moments are thus seen to be infinitely small increments and are analogous to the infinitesimals of Leibniz. In the application of this method to the solution of problems the final step always involved the rejection of infinitely small quantities as of no value in comparison with the other terms. Newton felt that the rejection of terms, no matter how small, always introduced a source of error in the results, and proceeded to build his theory of fluxions on a basis of rates, which involved a concept of limits.

In the introduction to the *Quadratura Curvarum* appeared an illustration of this second method of fluxions, in which he found the fluxion of x^n to be nx^{n-1}.[4]

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[1] A. DeMorgan, *Essays on the Life and Works of Newton*, Open Court Publishing Co., 1914, p. 25.

[2] F. Cajori, *A History of Mathematics*, The Macmillan Co., 1919, p. 193.

[3] Cajori, op. cit., pp. 194–195.

[4] G. H. Graves, *Development of the Fundamental Ideas of the Differential Calculus*, Mathematics Teacher, 1910, Vol. 3, p. 86.

"Let a quantity x flow uniformly and let it be required to find the fluxion of x^n. During the time in which x, by flowing, is becoming $x+o$, the quantity x^n becomes $(x+o)^n$; that is, by the method of infinite series,

$$x^n + nox^{n-1} + \frac{nn-n}{2}oox^{n-2} + \cdots$$

and the increments o and

$$nox^{n-1} + \frac{nn-n}{2}oox^{n-2} + \cdots$$

are to each other as 1 and

$$nx^{n-1} + \frac{nn-n}{2}ox^{n-2} + \cdots.$$

Now let the increment vanish and the last ratio will be 1 to nx^{n-1}."

In the same treatise Newton connected this idea of fluxions with the problem of tangents. He considered a tangent as the limiting position of a secant as may be seen from his statement: "If the points (of intersection with the curve) are distant from each other by an interval however small, the secant will be distant from the tangent by a small interval. That it may coincide with the tangent and the last ratio found, the two points must unite and coincide altogether."[1]

Newton always considered fluxions as finite velocities, which were determined by the ratios of nascent or evanescent quantities. Feeling the logical difficulties involved in such a concept, he attempted to overcome them by his notion of prime and ultimate ratios which he developed in the *Principia:* "The ultimate ratios in which quantities vanish, are not really the ratios of ultimate quantities, but limits toward which the ratios of quantities, decreasing without limit, always approach; and to which they can come nearer than any given difference, but which they can never pass nor attain before the quantities are diminished indefinitely."[2]

In contrast to Newton who had received his early mathematical training under such a capable mathematician as Isaac Barrow, Leibniz (1646–1716) was a self-taught mathematician. However, he had the good fortune to make the acquaintance of the eminent physicist, astronomer, and mathematician, Christian Huygens (1629–1695) who advised him to study the works of Pascal, Cavalieri, and Descartes.

While studying Cartesian geometry Leibniz became interested in the problems concerning direct and inverse tangents. In his approach to these prob-

[1] Graves, loc. cit.

[2] Smith, *A Source Book in Mathematics*, p. 618, translation from the *Principia* by Professor Evelyn Walker of Hunter College.

lems he drew the characteristic triangle, an infinitely small triangle included between differences of the ordinates, differences of the abscissae, and infinitely small arcs of the curve. In a tract written in 1673 he made two rather significant statements: "(1) There is a direct connection between the inverse method of tangents and quadratures; (2) The matter will be most accurately investigated by tables of equations."

By the side of, and linked closely with, the progress of the differential calculus of tangents was developed the science of integration. From an early conceived notion of the relation of quadratures to the "differences of applied lines," Leibniz arrived at a direct connection between them. Starting with the Cavalierian concept that areas are but the summation of lines, he first created $\int$, the symbol for summation (or integration). Reasoning that since summation increased the power, or dimension, so would differencing decrease the same, he progressed to the true meaning of differences and introduced the symbol, d, for the process of taking differences. Henceforth integration, in place of being that of which differentiation was the contrary calculus, became the inverse of differentiation.

In an undated manuscript, probably written about 1680, Leibniz showed the power of his new symbolism. Wallis, Pascal, and others had stated that areas might be obtained by the summation of an infinite number of rectangles whose altitudes were infinitesimals, but in their solutions had proceeded as if these rectangles were straight lines in arithmetic progression. Leibniz recognized the presence of this infinitestimal altitude by including it in the expression for an area. He wrote, "I represent the area of a figure by the sum of all the rectangles contained by the ordinates and the differences of the abscissae . . . I represent in my calculus the area of a figure by $\int y\,dx$, or the sum of the rectangles contained by each y and the dx that corresponds to it." Then he proceeded to link his method with that of Cavalieri by saying, "here, if the dx's are taken equal to one another, the method of Cavalieri is obtained."[1] Leibniz had arrived, in a manner far superior to any he had anticipated, at his second objective, namely that "the quadratures of all figures follow from the inverse method of tangents, and thus the whole science of sums and quadratures can be reduced to analysis, a thing that nobody even had hopes of before."[2] He had thus created a calculus dependent on the calculus of differences, which he first termed *Calculus Summatorius* and later *Calculus Integralis*.

The calculus of differences of Leibniz did not differ in principle from the method of tangents used by Barrow. There was, however, the important difference that whereas Barrow was interested primarily in geometry and applied his method only to that end, Leibniz was able to treat algebraic expressions by his method, geometry simply being one of the fields in which applications could

[1] J. M. Child, *The Early Mathematical Manuscripts of Leibniz*, Open Court Publishing Co., 1920, p. 138.

[2] Child, ibid., pp. 60–61.

be made. Each gave a set of rules for differentiation and integration, the former geometric, and the latter algebraic. Leibniz termed his set of rules an "algorithm." It was this algorithm, along with the notation for differentiation and integration, that marked his work as a distinct step forward in infinitesimal analysis.

A great dispute as to the priority of the discovery of the infinitesimal calculus arose between Newton and Leibniz. Friends of the two men took sides until finally the dispute became so severe that it influenced the development of the subject for many years to follow. For the most part, the mathematicians of England followed the Newtonian idea while those of the continent followed Leibniz's point of view.

On the continent, the two Bernoulli brothers, Jacques (1654–1705) and Jean (1667–1748), Marquis de l'Hôpital (1661–1704), and d'Alembert (1717–1783) were among the most prominent of the early contributors to the development of these newly proposed methods of infinitesimal analysis. In Great Britain a Scotchman by the name of Craig (1656–1718) published several treatises in the form of books and contributions to the *Philosophical Transactions* (London) in which he made use of the notation and method suggested by Leibniz. It is said that, in preparing one of these manuscripts for publication, he submitted it to Newton to read and that this manuscript was influential in causing Newton to allow his method of fluxions to be made public in the 1693 edition of Wallis's algebra.[1] The priority dispute influenced Craig to use the fluxional notation in his later papers. Other followers of Newton were De Moivre (1667–1754), Taylor (1685–1731), MacLaurin (1698–1746), and Cotes (1682–1716). With the exception of Taylor and MacLaurin they all used fluxions as infinitely small quantities, thus grafting Newton's notation onto the calculus of Leibniz.

The two methods were not without their critics. Most prominent among these were Nieuwentijt (1654–1718) and Rolle (1652–1719) on the continent and Berkeley (1684–1753) in Great Britain. Nieuwentijt's objections were somewhat philosophical, being; (1) the inaccuracy of rejecting infinitely small quantities; (2) the doubtful existence of differentials of higher order; and (3) the lack of universality of the method in that it had not been applied to other than algebraic expressions. Leibniz, in answer, skillfully met the first objection, only partially and rather poorly the second, and completely ignored the third.[2] Rolle's objections were largely computational and personal. Berkeley's objections were largely metaphysical, being inspired by the attitude of certain mathematicians who had complained of the incomprehensibility of religion. In his remarkable book, *The Analyst*, he attempted to show that the calculus was much more incomprehensible, basing his argument on what he considered to be four weaknesses of the subject, namely the dropping of terms, the concept of

[1] F. Cajori, *A History of the Conceptions of Limits and Fluxions in Great Britain from Newton to Woodhouse*, Open Court Publishing Co., 1919, pp. 38–39.

[2] Child, op. cit., p. 145.

velocity, the higher order of fluxions, and the ratio of evanescent increments. Cajori refers to the publication of *The Analyst* as "the most spectacular event of the century in the history of British mathematics."[1] Whatever may have been the intent of the author in this publication, the results were exceedingly beneficial to the calculus in that it provoked the clarification of some of its concepts.

We are indebted to Euler (1707–1783) not so much for his philosophy of the calculus, though he introduced a few new ideas with reference to the ratios of increments, but more for his two treatises *Institutiones Calculi differentialis* (1755) and *Institutiones Calculi integralis* (1768–1770). These two works, combined, form a complete text in which was summed up all that was then known on the subject, and the first text to treat the subject matter of the calculus in a general and systematic manner.

The principles of the calculus were, during the latter half of the eighteenth century, still involved in philosophical difficulties which were somewhat serious in nature. In the sub-title to his *Théorie des Fonctions Analytiques*, Lagrange (1736–1813) states that this work contains "the principles of the differential calculus, freed from any consideration of infinitesimals, of vanishing elements, of limits and fluxions, and reduced to the algebraic analysis of finite quantities." In this statement he has summed up the fundamental philosophical points of contention. In the introduction he proceeds to elaborate upon these difficulties, finally somewhat summarizing his statements in these words: "But this method (the method of fluxions) has, as that of limits of which it is only an algebraic translation, the great inconvenience of considering the quantities in the state, so to speak, in which they cease to exist, for, although we can conceive of the ratio of two finite existing quantities, that ratio does not offer to the mind a clear and precise idea of what happens when both terms become zero at the same time."[2]

Landen (1719–1790) an English geometrician, in a work entitled "The residual analysis a new branch of the Algebraic art" and published in London in 1764, proposed a purely analytic method to replace the method of fluxions. Landen proposed that "instead of the infinitesimal or null differences of varying quantities, different values of these quantities should be used, and they should be equated only after the vanishing factor had been caused to disappear through division."[3] Although this put the differential calculus on a more rigorous foundation, it caused the subject to lose a great deal of its simplicity and ease of operation.

Lagrange in commenting on his own contribution said "in a memoir . . . the purpose of which was to present the analogy between differentials and positive powers, and between integrals and negative powers, I stated that the theory

[1] Cajori, ibid., p. 57.

[2] J. L. Lagrange, *Théorie des Fonctions Analytiques*, Courcier, Paris, 1813, p. 4.

[3] Lagrange, loc. cit.

of development of functions in series contained the real principles of the differential calculus, freed from any consideration of infinitesimals or limits, and, by this method, I proved Taylor's theorem, which is fundamental to the theory of series, and which up to this time, had been established only by means of the calculus, or by the consideration of infinitesimal differences."[1] In this development of functions in series, Lagrange called the first term the *primitive function*, and the different functions of the same variable, which appeared as multipliers in subsequent terms of the series, he called *derived functions* since they depended only on the primitive function and not on the undetermined increment used to effect the expansion. He expressed surprise that Newton had missed this approach to the calculus, since it was he who had first made any progress in the study of series.

Though Lagrange's method of developing the calculus succeeded in abstracting the notion of function, mathematicians soon detected that his arguments were fallacious. He had sought to define his *derived functions* in terms of coefficients in infinite series, concerning the convergence of which he knew nothing, and his proof that $f(x+h)$ could always be expanded in a series of ascending powers of h was by no means rigorous.[2]

With the researches of A. L. Cauchy (1789–1857) we have the culmination of the efforts to place the methods of infinitesimal analysis on a sound metaphysical basis. He made the first rigorous attempt to establish Taylor's theorem and thus to substantiate Lagrange's efforts to abstract the notion of function and place the calculus on an analytic basis. He also introduced a new method of considering limits and a new theory of continuity of functions, each of which helped in the exposition of the fundamental principles of the calculus.

In the meantime the "Analytical Society" had been formed at Cambridge, England, in 1813 and through the untiring efforts of the society the Leibnizian or differential notation dy/dx was introduced to replace the Newtonian, or fluxional, notation $\dot{y}$.

Thus we see that by the end of the first quarter of the nineteenth century infinitesimal analysis had reached the point of development where its fundamental principles and methods were on a sound metaphysical basis, its notation fairly well systematized and standardized, and its simplicity and facility of operation quite generally appreciated. Varied and interesting were the avenues of thought that were to become open sesames to future mathematicians under the influence of this new powerful tool of mathematical research.

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[1] Lagrange, ibid., p. 5.

[2] Cajori, op. cit., p. 257.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A PROBLEM IN GEOMETRICAL PROBABILITY

By C. H. FISCHER, University of Minnesota

A problem[1] recently solved in this Monthly was stated as follows: A slender rod of length $2a$ rests on a circular table of radius r, $r>a$. What are the probabilities that neither end, one end, or both ends will project over the edge of the table?

As is usual in the case of problems of this type, the solution is dependent upon certain basic assumptions; in this case, upon the assumptions made as to the manner in which the rod was placed upon the table. The published solution was obtained under the assumptions that perhaps appeal most to one's intuition, namely, that the direction of the rod is immaterial, and that the probability that the center of the rod will lie in any given region is proportional to the area of that region. A further study of this problem was made because of its resemblance to the well known Bertrand's Paradox, which asks for the probability that a chord drawn at random across a circle be at least as long as the side of the inscribed equilateral triangle. Czuber[2] gives six illustrative solutions using different basic assumptions as to what shall be considered as "equally likely" in drawing a chord at random across the circle, and he points out that further solutions could thus be obtained, each correct upon the basis of its own hypotheses.

In the present problem the rod may be considered as lying along a chord of the circle. An additional variable is necessary here, since the position of the center of the rod along the chord must also be considered. The variables of the problem may be chosen in various ways. For example, the angle the rod makes with a fixed line in the plane may be selected as the first variable, and the coordinates of the center of the rod as the other two variables. Another possible selection might find the first point of intersection of the rod (produced) with the circle, chosen as one variable, the second intersection as the second variable, and the position of the center of the rod along this line as the third variable. Other selections could be made. Let us examine the results obtained under a few particular sets of assumptions, making use of probability functions instead of using purely geometric methods.

1a). First, consider the assumptions of the published solution, as cited above. Choose rectangular coordinates, with the circle given by $x^2+y^2=r^2$, and

[1] Problem 3539, this MONTHLY, vol. 40 (1933), pp. 60–61.

[2] Czuber, *Wahrscheinlichkeitsrechnung*, (1908), vol. 1, pp. 106–109.

the rod parallel to the x-axis. The probability function of the distance of the center of the rod from the y-axis is $4(r^2-x^2)^{1/2}/\pi r^2$. That of the distance of the rod from the x-axis is $(r^2-x^2)^{-1/2}$. Then the probability that neither end of the rod will project over the edge of the table is given by

$$\frac{4}{\pi r^2}\int_0^{(r^2-a^2)^{1/2}}\int_0^{(r^2-y^2)^{1/2}-a} dxdy.$$

The expressions for the other two probabilities may be set up in the same manner. The results, as published, are:

A). The probability that neither end of the rod will project over the edge of the table is:

$$\frac{2}{\pi}\arccos\frac{a}{r}-\frac{2a}{\pi r^2}(r^2-a^2)^{1/2}.$$

B). That one end of the rod will project:

$$\frac{4a}{\pi r^2}(r^2-a^2)^{1/2}-\frac{a}{\pi r^2}(4r^2-a^2)^{1/2}-\frac{4}{\pi}\arccos\frac{a}{r}+\frac{4}{\pi}\arccos\frac{a}{2r}.$$

C). That both ends of the rod will project:

$$\frac{a}{\pi r^2}(4r^2-a^2)^{1/2}-\frac{2a}{\pi r^2}(r^2-a^2)^{1/2}+\frac{4}{\pi}\arcsin\frac{a}{2r}-\frac{2}{\pi}\arcsin\frac{a}{r}.$$

1b). Now assume that the position of the first point of intersection of the rod (produced) and the circle is immaterial. Select one such point. Assume that the probability that the center of the rod will lie in any given region is proportional to the area of that region. Choose polar coordinates, with the circle given by $\rho=2r\sin\theta$. Then the probability function of the angle made by the rod and the x-axis is $4\pi^{-1}\sin^2\theta$. That of the distance of the center of the rod from the origin is $(2r\sin\theta)^{-1}$. Then the probability that neither end of the rod will project over the edge of the table is given by

$$\frac{2}{\pi r}\int_{\arcsin a/r}^{\pi/2}\int_a^{2r\sin\theta-a}\sin\theta\,d\rho d\theta.$$

The expressions for the other probabilities may be set up in the same manner. The results are identical with those of 1*a*, as the basic assumptions, though somewhat different in form, are essentially equivalent.

2). Assume that the direction of the rod is immaterial. Let the circle be given by $x^2+y^2=r^2$. Then, for any fixed direction, assume that all distances of the rod from the x-axis, chosen parallel to the rod, are equally likely. This probability function is given by $1/r$. For any fixed distance from the x-axis, assume further that all distances of the center of the rod from the y-axis are equally likely. This probability function is given by $(r^2-y^2)^{-1/2}$. Then the

probability that neither end of the rod will project over the edge of the table is given by

$$\frac{1}{r}\int_0^{(r^2-a^2)^{1/2}}\int_0^{(r^2-y^2)^{1/2}-a}(r^2-y^2)^{-1/2}dxdy.$$

It will be noticed that we are integrating over the same area as in 1*a*, but with a different integrand. The expressions for the other two probabilities may be set up in the same manner. The results, listed in the same order as those under 1*a*, are

$$A).\qquad \frac{1}{r}\left\{(r^2-a^2)^{1/2}+a\cdot\arcsin\frac{a}{r}-\frac{\pi a}{2}\right\}.$$

$$B).\qquad \frac{1}{r}\left\{2r+\frac{\pi a}{2}-2a\cdot\arcsin\frac{a}{r}-2(r^2-a^2)^{1/2}\right\}.$$

$$C).\qquad \frac{1}{r}\left\{(r^2-a^2)^{1/2}-r+a\cdot\arcsin\frac{a}{r}\right\}.$$

3). Assume that the position of the first point of intersection of the rod with the circle is immaterial. Let the circle be given by $\rho=2r\sin\theta$. Then assume that all positions of the second intersection are equally likely, or the equivalent of this, that all angles made by the rod and the x-axis are equally likely. The probability function of this angle is $2/\pi$. Then assume that for any fixed angle all distances of the center of the rod from the origin, measured along the chord, are equally likely. This probability function is given by $(2r\sin\theta)^{-1}$. Then the probability that neither end of the rod will project over the edge of the table is given by

$$\frac{1}{\pi r}\int_{\arcsin a/r}^{\pi/2}\int_a^{2r\sin\theta-a}\csc\theta\,\rho d\theta.$$

Here we are integrating over the same area as in 1*b*, but with a different integrand. The expressions for the other two probabilities may be set up in the same manner. The results, listed in the same order as those under 1*a*, are

$$A).\qquad 1-\frac{2}{\pi}\arcsin\frac{a}{r}+\frac{2a}{\pi r}\log\frac{r-(r^2-a^2)^{1/2}}{a}.$$

$$B).\qquad \frac{4}{\pi}\arcsin\frac{a}{r}-\frac{4}{\pi}\arcsin\frac{a}{2r}+\frac{2a}{\pi r}\log\frac{2r-(4r^2-a^2)^{1/2}}{a}-\frac{4a}{\pi r}\log\frac{r-(r^2-a^2)^{1/2}}{a}.$$

$$C).\qquad \frac{4}{\pi}\arcsin\frac{a}{2r}+\frac{2a}{\pi r}\log\frac{r-(r^2-a^2)^{1/2}}{2r-(4r^2-a^2)^{1/2}}-\frac{2}{\pi}\arcsin\frac{a}{r}.$$

Analytically, cases 2 and 3 are as simple as case 1. If probability is defined as the limit of relative frequency, one might inquire into the possibility of testing these solutions experimentally. If an attempt is made to design mechanical devices for placing the rod upon the table in such a manner that the assumptions of 1, 2, and 3, respectively, will hold, we find little difficulty in designing such a device for case 1. Considerable difficulty is encountered with the remaining two cases, however, and the device would be either quite complicated mechanically, or else involve a process of two steps. This investigation helps to demonstrate the reasonableness of the basic assumptions of the published solution.

It might be of some interest to compute the numerical values of these probabilities for some special cases. If $r=5$ and $a=4$, for example, these probabilities are, approximately, as given in the following table:

	Case 1	Case 2	Case 3
Probability that neither end projects	.1037	.0854	.0565
Probability that one end projects	.8014	.5726	.5649
Probability that both ends project	.0949	.3420	.3786

Numbers Equal to the Sum of the Digits of Their Nth Powers

By G. E. Raynor, Lehigh University

On page 457 of the first volume of Dickson's *History of the Theory of Numbers* one finds the following statement: "Moret-Blanc proved that 1, 8, 17, 18, 26, 27 are the only numbers equal to the sum of the digits of their cubes." The writer has amused himself by finding similar results for powers other than cubes. The following table gives the only numbers N which are equal to the sum of the digits of their nth powers, for values of n up to 8.

n	N
1	1, 2, 3, 4, 5, 6, 7, 8, 9
2	1, 9
3	1, 8, 17, 18, 26, 27
4	1, 7, 22, 25, 28, 36
5	1, 28, 35, 36, 46
6	1, 18, 45, 54, 64
7	1, 18, 27, 31, 34, 43, 53, 58, 68
8	1, 46, 54

We shall illustrate the method by which these results were obtained for the

case $n=4$. As is evident, our method is very crude. It may interest readers of the Monthly to find a more refined process for extending the above table.

Let m be the number of digits in N. Then

$$10^{m-1} \leqq N < 10^m \tag{1}$$

and hence $N^4 < 10^{4m}$. It follows that the number of digits in N^4 is not greater than $4m$ and hence that the sum of these digits is not greater than $9 \times 4m = 36m$. Now, by the first part of (1) $36m$ will be less than N if

$$36m < 10^{m-1} \tag{2}$$

and this in turn will be satisfied if

$$100m < 10^{m-1}$$

or

$$m < 10^{m-3}. \tag{3}$$

But $10 > e^2$, where e is the base of the natural system of logarithms, and hence if

$$m < e^{2(m-3)} = 1 + 2(m-3) + \cdots \tag{4}$$

(2) and (3) will be satisfied. But (4) is satisfied if

$$m \leqq 1 + 2(m-3)$$

which reduces to

$$m \geqq 5.$$

Also, by inspection we see that (2) is satisfied for $m=4$. Hence, it follows that a number which is equal to the sum of the digits of its 4th power cannot have more than three digits.

However, a number of three digits can have in its 4th power not more than twelve digits which can add up to not more $12 \times 9 = 108$. Hence $N \leqq 108$. But $110^4 = 146{,}410{,}000$ and it is evident that the number less than this whose digits have the greatest sum is 99,999,999, the sum of the digits being 72. Hence $N \leqq 72$. By a similar process we find that no number $\leqq 72$ can have a 4th power the sum of whose digits is more than 64. Now running down a table of fourth powers[1] from 64 we find that the numbers 1, 7, 22, 25, 28, 36 satisfy our condition and hence are the *only* numbers which are equal to the sum of the digits of their fourth powers.

In closing we wish to raise the following question. Given an arbitrary positive integer n, does there exist an integer N, other than unity, such that the sum of the digits of N^n is equal to N?

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[1] Glover's Tables of Applied Mathematics give the first eight powers of all integers from 1 to 100.

A Suggestion Regarding Foreign Languages in Mathematics

By E. T. Bell, California Institute of Technology

While following a mathematical discussion in which English, French, and German were used by various speakers, I was struck by the remarkably small number of common, non-technical words sufficient to lubricate the technical terms and to present a mathematical argument intelligibly. The common words include conjunctions, prepositions, personal pronouns, a few nouns, and a handful of verbs, particularly the auxiliaries. Some of the common verbs, like "put," or "take," of course are used with uncommon meanings, but these idioms could be included in the technical vocabulary. For an argument in analysis, algebra, or the theory of numbers, it would seem that 300 common words are more than sufficient; for geometry the number may be higher. For a philosophical discussion dealing, say, with mathematical logic, much more would be necessary. To the 300 or so common words must be added the technical vocabulary of the subject discussed.

It should be possible, by sampling pages of foreign mathematical books and periodicals to form a reasonable estimate of the number of *common* words necessary for a given branch of mathematics. If the guess of 300 is anywhere near the truth, it would be worth someone's time to compile such common vocabularies for the languages in which mathematics is alive. By concentrating on the minimum lists, supplemented by dictionaries if necessary, a working competence in two or three languages might readily be acquired by reading minutely two or three fairly long papers of which the mathematical subject matter was already familiar to the reader.

The suggestion that minimum lists be compiled is not an attempt to sabotage the usual teaching of French, German, and Italian. It is recognized that without a fair mastery of these languages, mathematical education soon comes to a halt. But what about Dutch, the Scandinavian languages, and Russian? Much Russian work in the theory of numbers, for example, is practically buried for years to most readers, and little of it ever receives adequate translation or review.

It is granted that such a hand to mouth method of gaining a mathematical reading knowledge of a foreign language would lack polish, to say nothing of syntax. But for those who are not born polyglots, some expedient is demanded. And it may be recalled that Sophus Lie, in spite of continued residence and lecturing in Germany, never did master the niceties of die, der, das. He even went so far as to compliment a rising young American mathematician, whose German was as efficient and ungrammatical as his own, on the clarity with which he expressed his ideas. A vocabulary is a necessity; syntax is a luxury (for mathematicians).

I believe that a study of this question, resulting in minimum common and technical lists, would be of value to students of mathematics. If the work were well done, the resulting pocket "dictionary" should find a publisher without difficulty. So there may be something in it, in more ways than one.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

College Algebra. By H. P. Pettit and P. Luteyn. New York, John Wiley & Sons, 1932. 276 pages, 2 tables. $1.90.

Practically all of the many college algebras that have been written during the past several years devote the early chapters of the text to a review of elementary algebra. A most interesting feature of this book is that it has broken away from this conventional practice and treatment. The authors believe "that it is a pedagogical mistake to start the course in college algebra with an extensive *review* of elementary algebra" because it is discouraging and fails to arouse interest in the minds of the students. Therefore in the first two chapters they introduce new material, namely, functions, graphs, and the summation convention, but they also incorporate a considerable amount of review of elementary algebra.

Apart from the arrangement, much the same material and treatment is found in this text as in other college algebras. Four separate chapters, III, VIII, XV, and XVIII, on problem solving only are included in the book. The reviewer is disappointed in finding that probability is omitted from this text. It not only naturally follows permutations and combinations, but it is a subject which generally appeals very much to the interest of the students. The binomial theorem follows permutations and combinations and is developed by means of combinations, although it is again treated later by mathematical induction. The method of completing the square in developing the formula for the solution of the general quadratic equation differs from that usually found in college algebras. It is also interesting to note that no particular quadratic equations are solved by the method of completing the square before the development of the formula for the general quadratic by this method. Probably the latter would be better understood by the student if he applied the method first to a few particular equations. The introduction of the summation convention is a valuable addition to college algebra. Relatively a great deal more space (40 pages) is devoted to integral and fractional exponents, radicals, and exponentials, than is usually found in college algebras. This is a noteworthy feature of the book, and the student should complete the course with a better than average knowledge of that particular phase of college algebra.

Apparently a special effort has been made by the authors to connect the subject with the mathematics which is to follow; and, in the reviewer's opinion, the strongest feature of the text is the inclusion of numerous problems which not only have practical value but are very closely related to the other sciences. There are many problems dealing with automobiles, airplanes, motorboats,

taxes, stocks, investments, rates, mechanics, heat, and electricity, which should be of great interest to the student. Problems related to physics and chemistry are very frequently introduced. Possibly a greater use of bold faced type to emphasize the important theorems, etc., would add value to the book.

On the whole it is well written, and the student using it as a text should not only get a good knowledge of college algebra but at the same time learn much about the physical sciences with which mathematics is most closely related.

E. D. WELLS

A Short Course in Trigonometry. By James G. Hardy. New York, The Macmillan Company, 1932. ix+181 pages: and the *Macmillan Tables*, revised edition, edited by E. R. Hedrick, 1930, xx+142 pages. $2.25

The arrangement of the material in this trigonometry differs from the usual sequence in that the chapter on logarithms and logarithmic solution of triangles is relegated to a subordinate position which, in this instance, is the last chapter. This, we believe, is greatly to be desired.

It is the opinion of the reviewer that the author has achieved his professed aim of making "clear why each topic is taken up" by the care with which they are introduced. It should interest the thoughtful student to see ***in the book*** the meaning of such terms as "co-function" and "double-angle" of formula fame. The instructor should welcome the insistence upon non-memorization of those results easily obtained by the application of a few simple fundamental concepts.

The book has been carefully written. This is evidenced by the discussion of such troublesome points—to the student—as why $-x$ does not necessarily mean that x is negative, and the meaning of the symbol $\tan 90° = \infty$.

A generous supply of exercises has been provided and some answers. A feature of the answers is that it is always indicated whether four or five place tables have been used in the computation. The tables are more numerous than those usually found in textbooks on trigonometry, for they include tables of powers, roots, reciprocals and many others, fourteen in all.

We believe the book to be eminently teachable and one that the student can read with understanding.

C. A. NELSON

Mathematics of Relativity (Lecture Notes). By G. Y. Rainich. Ann Arbor, Edwards Brothers, Inc., 1932. 67 pages.

Professor Rainich's book should prove welcome both to those who have occasion to give a course in Relativity and to those who are being initiated into the subject. This work, as the title indicates, is patterned after the lecture notes of the author. It is, however, complete as a presentation of the mathematics of Relativity in that it offers a background adequate for independent work in the subject.

The book is divided into five chapters. The main purpose of the opening chapter is to build up on the assumptions of classical physics the complete ten-

sor. Here the notion of tensor is used only to identify a group of associated physical quantities. At the close of the chapter the central question is raised: do the differential relations satisfied by the complete tensor determine the fundamental physical quantities involved? Toward answering this question the remainder of the book is devoted. The second chapter opens with some fundamental facts of four dimensional geometry followed by a formal presentation of its axiomatic basis leading to the author's own "vector argument" treatment of tensors. The author then considers the invariance of the equations of physics, the role of the imaginary fourth coordinate, and the representation of curves in a coordinate system moving with respect to a given system. The third chapter is devoted to Special Relativity (a partial answer to the question of Chapter 1), attention being given to generalized mass, the Lorentz transformations, addition of velocities, motion of photons, and the field of an electron in motion.

In the fourth chapter, which is given to curved space, the central role of the Riemann tensor in its identification with the complete tensor is brought out. The representation of tensors in general co-ordinates, and the physical significance of covariant and contravariant representations as well as differentiations are here treated. At the close of the chapter geodesics are defined in curved space. The final chapter opens with a clear and concise analysis of the task of General Relativity and proceeds to answer further the question put in the first chapter. It presents in closing the three important results of General Relativity which have been verified by experiment, namely the precession of the perihelion of a planet, the effect on the path of light of the gravitational field through which it passes, and the "Red Shift" in a Solar Spectrum.

This work shows considerable care in preparation, the subject matter being organized into an organic whole following a plan outlined in the introduction. The author is to be thanked for having added to his original contributions this exposition of the mathematics of Relativity, which in virtue of the arrangement and selection of its subject matter, and the clarity and conciseness of its treatment, is the most readable text on the subject that has come to the attention of the reviewer. It is to be hoped that these lithotyped notes will be succeeded in the next edition by a printed volume.

S. B. LITTAUER

Freshman Mathematics. By H. L. Slobin and W. E. Wilbur. New York, Ray Long and Richard R. Smith, Inc., 1932. xviii+438 pages. $2.60

This text covers in three distinct parts the subjects of college algebra, trigonometry, and analytic geometry. The space devoted to these is almost exactly in the ratio 2:1:2. This arrangement should appeal to those who

(i) desire a single text book for the entire year,
(ii) prefer a more or less traditional topical arrangement,
(iii) believe that calculus should be postponed until the second year.

On these points there is, of course, a considerable division of opinion. We do not mean to say that the authors have not achieved a certain degree of unifica-

tion. Thus graphing in rectangular coordinates is first introduced in the study of the linear function in algebra. It is continued in the chapter on the quadratic function and again in the solution of simultaneous equations. The trigonometric functions are graphed in the section on trigonometry. The work in conic sections is definitely tied up with the chapter in algebra on the quadratic function, although separated from it by some two hundred pages.

Nearly half of the section on algebra is a review of school algebra. This is followed by chapters on the usual array of disconnected topics, logarithms, progressions, permutations and combinations, probability, mathematical induction, partial fractions. The function concept is emphasized. Functions are graphed as a means of exhibiting their behavior. The authors do not find the zeros of the quadratic function so that they can draw its graph. Rather, the graph is used as a means of determining the nature and even the magnitude of the zeros. (Incidentally the authors use root for zero.) This is a distinct advance over many existing texts. Students of calculus who regard the first and second derivatives chiefly as aids in curve sketching can hardly be said to have grasped the real significance of the subject. Some readers will not like the expression, "the function does not cut the x-axis" (p. 59).

The part on trigonometry does not require any special comment. As usual the student is asked to prove certain identities. The reviewer thinks that the purpose of this work would be far better served if the problem were to simplify given trigonometric expressions. This would not only develop initiative on the part of the student but prepare him for what he will later be called on to do.

Perhaps the greatest novelty of treatment occurs in the part on analytic geometry. The reader will look in vain for a chapter on the ellipse starting with the usual definition. He will find instead that the ellipse has been defined in the algebra as the curve whose equation is so and so. The other conic sections are similarly defined. There is a chapter on the properties of the conic sections.

The authors have included a chapter on empirical curves. This is preceded by a chapter which discusses the graphs of the needed functions. The book closes with a 22 page chapter on Solid Analytic Geometry.

There are numerous problems and answers are given to nearly all of them. We have noted very few misprints. The authors are careless in the matter of significant figures. We find for example on page 136, $\log_e 763 = x$, $763 = e^x = 2.71828^x$, $\log_{10} 763 = x \log_{10} 2.71828$, $2.8825 = x(.4343)$, $x = 6.6370$. Thus in one calculation 4, 5, and 6 significant figures are used. (Actually $\log_e 763 = 6.6373$ to the nearest fourth decimal place.)

Your reviewer thinks this text book should prove satisfactory for a course covering the subjects which it treats. It is our personal opinion, however, that some of the material ordinarily covered in algebra and analytic geometry should be reserved for later courses in these subjects. This would make it possible to devote a part of the freshman year to calculus.

C. A. SHOOK

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscripts should be typewritten, with double spacing, and with margins at least one inch wide. All club activity manuscripts for the academic year 1932–1933 should be submitted for publication not later than June 1, 1933.

CLUB ACTIVITIES

1931–32

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are engaged in study and research and the exchange of ideas in the field of mathematical science.

Pi Mu Epsilon of the University of Washington

Although we have not been a chapter of Pi Mu Epsilon very long, this being our first annual report, our organization is one of three years' standing, having been organized in the fall of 1929 by a group of graduate students in the Mathematics Department. The purpose of this group of students, who called themselves the Junior Mathematics Club to distinguish it from the Faculty Research Club, was to give the members opportunity to present papers on various phases of Mathematics in which they are interested, and to acquaint the other members with these various branches.

The plan in regard to topics presented has been one of permitting the speaker his choice of topic, and the result has been extremely interesting and instructive, a variety of subjects having been chosen and offered to the group. Meetings are held every other week, and the members take turns being responsible for the programs. Visitors are welcomed at these meetings, so every audience is composed of interested undergraduates together with the members.

The meetings and programs were as follows:

October 20, 1931: "Problems in the calculus of variations with discontinuous solutions" by Horace C. Ayres.

November 10, 1931: "Discussion of the brachistochrone by the calculus of variations" by John A. Carlson.

November 24, 1931: "Invariants of a quadratic by tensor algebra" by Philip N. Carpenter.

December 8, 1931: "D'Alembert's theorem" by Helen M. Copenhagen.

January 18, 1932: "Solutions of the equation $x^y = y^x$" by Mary E. Haller.

February 11, 1932: "Some particular applications of mathematics to economics" by Myra Jensen.

February 25, 1932: "Selections from the calculus of variations" by Kenneth Knox.

March 10, 1932: "Invariants of projective differential geometry" by Earl Rex.

March 31, 1932: "Fourier series in engineering" by Archie Adams.

April 20, 1932: "Cubic curves" by Thomas Doyle.

May 4, 1932: "Coordinate systems—generalized coordinates" by Ingomar Hostetter.

May 18, 1932: "The theory of the harmonic analyzer" by John Carlson.

June 2, 1932: "A problem in the calculus of variations" Master's thesis by Kenneth Knox.

The installation of this club as a chapter of Pi Mu Epsilon took place March 5, 1932, with Dr. Lee McFarlan who was formerly a member of the chapter at the University of Missouri acting as installing officer. After the installation ceremony, a banquet was held in honor of the occasion; Dr. R. E. Moritz, head of our department, and Dr. R. M. Winger, Professor of Mathematics, were our guests, the latter presenting a delightful talk on "Mathematics and Idealism."

This year thirteen new members have been elected, six being initiated on April 6, 1932, and the remaining seven on June 2, 1932. The total membership in school at present is twenty-one. All of the members are faculty members or graduate students in Mathematics, with a few undergraduates who are exceptionally capable in their work. The policy of the group has always been to stress the scholastic side of the fraternity, and to minimize details of the organization, only such time as is absolutely necessary for efficiently carrying out our program being spent on business.

The officers for the Spring Quarter of the year were: Kenneth E. Knox, Director; Mary E. Haller, Vice Director; Helen M. Copenhagen, Secretary and Treasurer. The officers are elected every quarter. On Decoration Day, the group joined with the undergraduate Mathematics Honorary in a picnic at one of the parks near Seattle.

HELEN M. COPENHAGEN, *Secretary*

LOCAL MATHEMATICS CLUBS

The Mathematics Club, Delta Chi, of the University of Toledo

This club was organized three years ago in order that the students might have an opportunity to consider various topics which, although closely related to the classroom work, are not generally included in the standard courses. Also it endeavors to create a desire for individual study and research. All students who have taken or are taking calculus are eligible for membership. This year there were forty-six members.

At the regular meetings, which are held the third Thursday of each month, a student discusses and demonstrates some phase of mathematics which is of interest to the entire group. At the close of the meetings refreshments are served.

Professor Wayne Dancer is Faculty Advisor for the club. The officers for the year were: Mary Krepleever, President; Fern Walker, Vice-President; Zora Powlesland, Secretary-Treasurer.

The meetings and programs were as follows:

September 26, 1931: Steak roast at Maumee Canal Side-Cut Park. This served as a reunion for the old members and a welcome for the new.

October 8, 1931: "Rhombic dodecahedrons" by Edwin Strutton.

November 12, 1931: "Mathematical puzzles and curiosities" by Professor Wayne Dancer.

December 18, 1931: Christmas party at the home of Professor J. B. Brandeberry. The program was: "Line coordinates" by Kenneth Rossman. We also enjoyed a mathematical treasure hunt and other games and the exchanging of gifts.

January 21, 1932: "The history of Pi" by Merrill Seps; Contest on necessary and sufficient condition.

February 25, 1932: "Continued fractions" by Edith Ein; Roll Call—each member answering with the name and accomplishments of some mathematician whose name began with the same letter as his own.

March 17, 1932: St. Patrick's Day Party (Open meeting). The program was: "Various proofs of the theorem of Pythagoras" by Jane Kamke; Banjo Solos by Kenneth Rossman; Mathematical contests.

April 21, 1932: "Singular points" by Enzia Parks.

May 20, 1932: "Configurations" by Professor Dancer; "Applications of the sine curve" by Professor Brandeberry; Third annual banquet at L'Aiglon; Installation of new officers.

FERN WALKER, *Vice President*

The Mathematics Club of Wellesley College

During the year 1931–1932 the Wellesley College Mathematics Club was organized under the direction of the following officers: Claudia E. Jessup, President; Emily A. Neal, Vice-President; Ruth S. Ball, Treasurer, Barbara Alden, Secretary, Dorothy Reinman, Junior Executive, and Professor Mabel M. Young, faculty advisor. The club officers for the following year are elected by

a majority vote at the last meeting of the club held in April or May. They are elected by the club members, and any undergraduate, except Freshmen, interested in mathematics may become a member simply by paying the club dues of seventy-five cents.

The officers are nominated in conference by the present club officers, but the President and Vice-President are always selected from members of the coming Senior class, and the Junior Executive from the coming Junior class. The faculty advisor to serve for the following year is formally asked by the out-going President, but the office succeeds each year to a different member of the faculty. There were seventy-two active members enrolled in the club for the year 1931–1932.

The aim of the club is to stimulate the interest of students for mathematics by providing entertainment through mathematical recreations, the reading of mathematical papers on selected topics, mathematical songs and so on. This year six meetings were held, the dates arranged by the committee on the social schedule of the college. The programs were as follows.

The first meeting, October nineteenth, was a "get-acquainted" meeting for the benefit of the new members. An ingenious arrangement of mathematical equations was worked out by which each person was given a slip of paper with a component of an analytic equation written on it, and he was told to find the group in which his component belonged. Then the members of their respective equations introduced themselves to the rest of the club members. Professor Helen A. Merrill gave a survey report of the history of the Mathematics Club, and mathematical songs were sung.

The second meeting was held November twentieth. Claudia Jessup opened the meeting with a numerical problem by which she could tell the age of any member present, all of whom were satisfactorily mystified. Topics were then given by the club officers. Barbara Alden spoke on the *Impossibility of Trisecting an Angle;* Emily Neal explained *Pascal's Triangle* and demonstrated the formation of magic squares, Ruth Ball proved very convincingly that one equals zero, and two equals one, Dorothy Reinman entertained the club with a varied repertoire of mathematical puzzles, among which she proved that anyone's age was equal to Methuselah's, and that the diagonal of a one-hundred foot square was two hundred feet.

The third meeting of the club was held January twenty-second when a demonstration of some new mathematical models recently acquired by the Mathematics Department was conducted by members of the faculty. The models were made of strings to form ruled surfaces of the second order. Curves were made visible by a lantern which showed the intersection of other curves on lantern slides with the string models. The demonstration was interesting as well as instructive.

For the fourth and fifth meetings, February nineteenth and March eighteenth, a Mathematics Club contest was announced in which the club members could compete for excellence in giving certain topics selected by the club officers. A prize of a book entitled *Mathematical Wrinkles* was given to the winner.

At the first of the two meetings Edythe Fairbanks spoke on *Spirals in Art and Nature;* Constance Wall on *Infinity in Geometry* and Elizabeth Richardson on *Maria Agnesi and the Witch Curve Named After Her.* The club members voted in closed ballots for the speaker they enjoyed most, and judged on the four points: poise, clearness and correlation, the topic's interest in general, and its interest for mathematics students.

At the second meeting topics were given by Ann Dunham on *Lewis Carroll;* Mary Dean Clement on *Familiar Curves in Architecture* and Persis Bullard on *Modern Plane Geometry and Some of Its Properties.* The prize as announced at the last meeting of the club—the supper meeting by tradition—was awarded to Elizabeth Richardson.

The supper meeting was held April fifteenth and supper was served by the club officers. Then the officers for the year 1932–1933 were elected.

At most of our meetings we found that the singing of mathematical songs provided amusement and created an informal atmosphere which was very pleasant. Professor Merrill has made up several songs of her own and presented them to the Club, and one of the Mathematics Club

members, Barbara Trask, collaborated with Miss Merrill in making up the air and melody for a song which was adapted to words from a poem by Arthur Guitterman.

We have had a large attendance this year, an average of over half our members being present at the meetings.

BARBARA ALDEN, *Secretary*

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, H. L. OLSON, AND WM. FITCH CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to Wm. Fitch Cheney, Jr. Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 36. *Proposed by B. H. Brown, Dartmouth College.*

Show that the thirteenth of the month is more likely to be Friday than any one of the other days of the week.

E 37. *Proposed by Arthur Haas, Thomas Jefferson High School, N. Y.*

In the following multiplication of a three-place number by a two-place number, all but three of the sixteen digits have been replaced by the letter x. Reconstruct the problem and show that the solution is unique.

$$\begin{array}{cccccc} & & x & x & x & \\ & & & x & x & \\ \hline & & x & x & x & \\ & x & x & 4 & & \\ \hline x & x & x & 0 & 1 & \end{array}$$

E 38. *Proposed by J. R. Musselman, Western Reserve University.*

It is well known that the midpoints of the sides of any plane quadrilateral constitute the vertices of a parallelogram. Determine the most general conditions under which the parallelogram becomes (a) a rhombus, (b) a rectangle, and (c) a square.

E 39. *Proposed by W. R. Ransom, Tufts College.*

Obtain both roots of the equation

$$x^2 - 365.04x + 45.04 = 0$$

correct to four significant digits, by means of four-place logarithm tables.

E 40. *Proposed by Maud Willey, Long Beach, Mississippi.*

Let $C_i=0$, $(i=1, 2, 3)$ be the equations of three circles. Prove that the three circles, $\sum_{i=1}^{3}K_{ij}C_i=0$, $(j=1, 2, 3)$, have the same radical center as the three circles $C_i=0$.

Prove that the four spheres, $\sum_{i=1}^{4}K_{ij}S_i=0$, $(j=1, 2, 3, 4)$, have the same radical center as the four spheres whose equations are $S_i=0$, $(i=1, 2, 3, 4)$.

Prove that the n hyperspheres, $\sum_{i=1}^{n}K_{ij}S_i=0$, $(j=1, 2, \cdots, n)$ in space of $n-1$ dimensions, have the same radical center as the n hyperspheres $S_i=0$, $(i=1, 2, \cdots, n)$.

E 41. *Proposed by V. F. Ivanoff, San Francisco.*

A variable circular arc of constant length, l, has one end fixed in position and direction. Find the locus of its other end.

SOLUTIONS

E 16. [1933, 51] *Proposed by G. A. Yanosik, New York University.*

Prove that the envelope of the circles whose diameters run from points on a parabola to its focus, is the straight line tangent to the parabola at its vertex.

I. *Solution by Mannis Charosh, New Utrecht High School, Brooklyn, N.Y.*

Let the equation of the parabola be $y^2=8ax$. The coordinates of the focus F are $(2a, 0)$. The problem is now equivalent to proving that each of the given circles is tangent to the y-axis.

If the point P $(2b, 2c)$ is any point on the parabola, then $c^2=4ab$. The coordinates of M, the midpoint of FP, are $(a+b, c)$. The equation of the circle centered at M and with radius FM is therefore

$$\begin{aligned}[x-(a+b)]^2+[y-c]^2 &= [a-b]^2+c^2\\ &= [a-b]^2+4ab\\ &= [a+b]^2.\end{aligned}$$

If $x=0$, the resulting equation has the double root, $y=c$, so that this circle is tangent to the y-axis for each point P on the parabola.

Since each circle goes through the focus F, it too is part of the envelope.

II. *Solution by Simon Vatriquant, L'Athénée Royale D'Ixelles, Bruxelles.*

Let F be the focus of the parabola, V the vertex, and P any point on the curve. Let the tangents at P and V meet at T, which is thus the pole of VP. Then the diameter through T bisects VP, and also bisects FP at M (since it is parallel to the base VF of the triangle PVF). Since the projection of the focus of any parabola upon any tangent to that parabola, is a point on the tangent at the vertex, we know that FT is perpendicular to TP. Then the circle on the diameter FP is centered at M, goes through T, and is tangent to VT, since VM is a diameter of the parabola. Since the tangent to the parabola at V is thus

tangent to every one of the designated family of circles, it forms an envelope of that family.

Also solved by Theodore Lindquist, C. C. Richtmeyer, J. E. Thompson, R. N. Walter, and the proposer.

E 17. [1933, 51] *Proposed by Morgan Ward, California Institute of Technology.*

In Bierens de Haan's "Nouvelles Tables d'Intégrales Définies," Table 113, formula 1, page 162, it is stated that

$$-\int_0^1 \frac{\log x dx}{1+x+x^2} = \frac{2\pi^2}{27}.$$

Show that this result is incorrect.

Solution by the proposer.

Since

$$\frac{1}{1+x+x^2} = 1 - x + \frac{x^3}{1+x+x^2},$$

$$-\int_0^1 \frac{\log x dx}{1+x+x^2} = -\int_0^1 (1-x)\log x dx - \int_0^1 \frac{x^3 \log x dx}{1+x+x^2}$$

$$> -\int_0^1 (1-x)\log x dx,$$

because $x^3 \log x/(1+x+x^2)$ is negative in the range of integration considered. Therefore

$$-\int_0^1 \frac{\log x dx}{1+x+x^2} > -\int_0^1 \log x dx + \int_0^1 x \log x dx = 1 - \tfrac{1}{4} = .75$$

while $2\pi^2/27 < .74$.

Mr. H. S. Uhler of Yale University points out that there was published many years ago a monograph entitled

Kongl. Svenska Vetenskaps-Akademiens Handlingar.
Bandet 24 N:o 5.
Examen des Nouvelles Tables d'Intégrales Définies
de
M. Bierens de Haan
Amsterdam 1867.
par
C. F. Lindman.

and in this monograph, on page 61, there appears

"Tab. 113.
1 à 4 sont fautives. Lisez

$$1. \qquad \int_0^1 \frac{\log x dx}{1+x+x^2} = -\frac{8}{27\sqrt{3}} H(\tfrac{1}{3}) = -0.7813024129."$$

Also solved by D. C. Duncan, Raymond Garver and J. B. Meyer.

E 19. [1933, 51] *Proposed by W. F. Cheney, Jr., Connecticut State College.*

Find the radius of a sphere whose surface area and volume are each numerically equal to π times a four-place integer, and show that the solution is unique.

Solution by Raymond Garver, University of California at Los Angeles.

Since $4R^2$ and $4R^3/3$ are each integers, R is obviously rational, with a denominator of 1 or 2, since $4R^2$ is an integer. Since $4R^3/3$ is also an integer, R is an integer divisible by 3. Since $999 < 4R^2$, $15 < R$. Since $4R^3/3 < 10000$, $R < 20$. But the only integer multiple of 3 between 15 and 20 is 18, so $R = 18$. For $R = 18$, the surface area is 1296π and the volume is 7776π, which in each case is a four-place integer times π.

Also solved by W. R. Ransom, Simon Vatriquant, E. E. Whitford and the proposer.

Advanced Problems

Send all communications about Advanced Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3612. *Proposed by H. Grossman, New York.*

Prove that the ratio of the sum of the hth, $(h+k)$th, $(h+2k)$th, etc. coefficients to the sum of all the coefficients in the expansion of $(a+b)^n$ converges to the limit $1/k$, as n approaches ∞, where $h = 1, 2, 3, \cdots, k$; and k is a positive integer.

3613. *Proposed by V. F. Ivanoff, San Francisco, Calif.*

Prove that

$$(R_x \cos\theta)^{2/3} + (R_y \cos\phi)^{2/3} + (R_z \cos\psi)^{2/3} = 2R^{2/3},$$

where R is the radius of curvature of a given curve at the given point (x_1, y_1, z_1); R_x, R_y, R_z are the radii of curvature of the projections of this curve on the coordinate planes YOZ, XOZ and XOY at the points (y_1, z_1), (x_1, z_1) and (x_1, y_1),

respectively; and θ, ϕ and ψ are the angles between the binormal to the curve and the coordinate axes.

3614. *Proposed by Leverett Davis, Jr., Seattle, Washington.*

If

$$f(x+y) = f(x)[1 - f(y)^2]^{1/2} + f(y)[1 - f(x)^2]^{1/2},$$

what conditions must be satisfied so that $f(x)$ will be the ordinary sine function? In other words, under what conditions will the law of addition define the sine?

3615. *Proposed by B. D. Roberts, New Mexico Normal University.*

A dust storm contains particles of two kinds identical except as to color, brown and yellow particles existing in the ratio of 3:2. If five particles of this dust enter my eye at random, determine the probability that two of them are brown and the other three are yellow.

SOLUTIONS

3340 [1928, 378]. *Proposed by the late V. M. Spunar, Chicago, Illinois.*

Solve

$$\begin{vmatrix} x^2dx, & y^2dy, & z^2dz \\ dx, & dy, & dz \\ x, & y, & z \end{vmatrix} = 0$$

consistent with the equation $xyz = 1$.

A Note by Otto Dunkel. The differential equation of the problem determines the lines of curvature of the given surface. For at a point P (x, y, z) of the surface the vector (x^{-1}, y^{-1}, z^{-1}) is normal to the surface; and at a neighboring point P' in the direction (dx, dy, dz) the vector $(x^{-1}-x^{-2}dx, y^{-1}-y^{-2}dy, z^{-1}-z^{-2}dz)$ is also normal to the surface, except for terms of higher order. The condition that the two normals ultimately intersect is the vanishing of the triple scalar product of the above three vectors, or, what is the same thing, the vanishing of the determinant formed from the components of the three vectors. The resulting equation reduces to that of the problem.

This problem is proposed in *A treatise on differential equations*, Forsyth, 2nd. ed. 1888 (Macmillan), p. 265, ex. 4. A solution is given in the translation into German by Maser, *Lehrbuch der Differential-Gleichungen*, 1889 (Vieweg), p. 626. See also *Solutions of examples in a treatise on differential equations*, Forsyth, 1918 (Macmillan), p. 117.

The last reference states that the differential equation is that of the lines of curvature of the surface $xyz = 1$; and that the integral to be associated with the given equation of the surface is

$$(1) \qquad (x^2 + \omega y^2 + \omega^2 z^2)^{3/2} + (x^2 + \omega^2 y^2 + \omega z^2)^{3/2} = A.$$

This may be verified in an indirect manner as follows: Set the first parenthesis

in (1) equal to U, and the second equal to $\bar{U}$. Since ω and ω^2 are the imaginary cube roots of unity, U and $\bar{U}$ are conjugate imaginaries when x, y, z are real; therefore A is a real constant. Set

$$
\begin{aligned}
&X_1 = U^{1/2} + \bar{U}^{1/2} \qquad X_2 = U^{1/2} - \bar{U}^{1/2}, \\
(2)\qquad &Y_1 = \omega U^{1/2} + \omega^2 \bar{U}^{1/2}, \quad Y_2 = \omega U^{1/2} - \omega^2 \bar{U}^{1/2}, \\
&Z_1 = \omega^2 U^{1/2} + \omega \bar{U}^{1/2}, \quad Z_2 = \omega^2 U^{1/2} - \omega \bar{U}^{1/2}.
\end{aligned}
$$

Consider the three surfaces

$$
(3)\qquad U^{3/2} + \bar{U}^{3/2} \equiv X_1 Y_1 Z_1 = A, \quad U^{3/2} - \bar{U}^{3/2} \equiv X_2 Y_2 Z_2 = B, \quad xyz = C,
$$

where A, C are real constants and B is a pure imaginary constant. At a point common to these three surfaces the three vectors with the components

$$
(4)\qquad \begin{matrix} xX_1, & yY_1, & zZ_1, \\ ixX_2, & iyY_2, & izZ_2, \\ yz, & xz, & xy, \end{matrix}
$$

are normal to the respective surfaces. It may be more convenient to obtain the first two sets from the U form of the equations of the surfaces. These vectors form a mutually orthogonal system. For in the case of the first and third, the scalar product is $xyz(X_1+Y_1+Z_1)$, and this last factor is clearly zero, since $1+\omega+\omega^2=0$ and (2) shows then that the parenthesis is zero. A similar reason applies to the second and third. For the first and second, we have for the scalar product, neglecting the factor i,

$$
\begin{aligned}
&x^2X_1X_2 + y^2Y_1Y_2 + z^2Z_1Z_2 = \\
&\qquad x^2(U - \bar{U}) + y^2(\omega^2 U - \omega \bar{U}) + z^2(\omega U - \omega^2 \bar{U}) = \\
&\qquad (\omega - \omega^2)\left[x^2(y^2 - z^2) + y^2(z^2 - x^2) + z^2(x^2 - y^2)\right] = 0.
\end{aligned}
$$

Hence any pair of these surfaces cut the third in its lines of curvature. Thus the first two in conjunction with the third are solutions of the differential equation of the problem, where C is there taken as unity.

3513. [1931, 461]. *Proposed by L. S. Johnston, University of Detroit.*

Given

$$
f(s, p) = \sum_{k=1}^{p} (-1)^{p+k}(2k - 1)_sH_{p-k}
$$

where

$$
{}_sH_r \equiv \frac{s(s+1)(s+2)\cdots(s+r-1)}{r!}, \qquad {}_sH_0 \equiv 1.
$$

Prove: (a) If p is odd, then $f(s, p)$ is positive for all values of s;

(b) If p is even, then $f(s, p)$ is positive, equal to zero, or negative according as s is less than, equal to, or greater than 3.

II. *Solution by Frank Ayres, Jr., Dickinson College.*

It will be shown in this solution that the statement of the problem is true for all real values of s. It has already been shown in a former solution [1933, 117] that it is true for all real values of s except in the following intervals in which the sign of $f(s, p)$ was not then determined:

(1) $\quad p = 2m + 1, \quad 4 - 2j < s < 5 - 2j, \quad j = 1, 2, \cdots, m,$

(2) $\quad p = 2m, \qquad 3 - 2j < s < 4 - 2j, \quad j = 1, 2, \cdots, m - 1.$

It will now be shown that $f(s, p)$ is positive in these intervals. In the previous solution it was shown that

$$f(s, p) = (-1)^{p-1} \sum_{k=1}^{p} (-1)^{k-1} k_{s-1} H_{p-k}. \tag{3}$$

If the terms with positive sign in the summation (except the last for p odd) are transformed by the relation $r\ {}_sH_r = (s+r-1)\ {}_sH_{r-1}$ there results for p odd

$$f(s, p) = p + (-1)^{p-1} \sum_{k=1}^{q-1} \frac{(2k-1)(s-1) - p}{p - 2k + 1} {}_{s-1}H_{p-2k}, \tag{4}$$

where q is the greatest integer in $(p+1)/2$. The expression (4) shows that $f(s, p)$ is positive in the intervals of (1) for which $j = 2, 3, \cdots, m$. The case $j = 1$ will be considered later.

For p even the first term p in (4) is dropped and the upper limit $q-1$ is replaced by q; and the resulting expression shows that $f(s, p)$ is positive in all the intervals of (2).

We shall now use the development of the previous solution

$$f(s, p) = (-1)^{p-1} \sum_{k=1}^{q} k_{s-3} H_{p-2k+1}. \tag{5}$$

Then

$$f(s, p + 2) = f(s, p) + (-1)^{p-1} \sum_{k=1}^{q+1} {}_{s-3}H_{p-2k+3}, \tag{6}$$

as is easily seen by combining like terms on the right in the two developments. It will be shown that the second term on the right is positive for p odd and $2 < s < 3$. Since $f(s, 1) = 1$, it will then follow by repeated use of (6) that $f(s, p)$ is positive for all odd values of p and for $2 < s < 3$. We have

$$\sum_{k=1}^{q+1} {}_{s-3}H_{p-2k+3} = \sum_{k=1}^{p+2} (-1)^{k+1}\ {}_{s-2}H_{p-k+2}, \tag{7}$$

by making use of the relation ${}_{s-3}H_n = {}_{s-2}H_n - {}_{s-2}H_{n-1}$. In the summation on

the right the terms with the minus sign are now transformed by means of $r\,{}_sH_r=(s+r-1)\,{}_sH_{r-1}$ and there results the equivalent expression

$$(8)\qquad {}_{s-2}H_{p+1}+(3-s)\sum_{k=1}^{q}\frac{{}_{s-2}H_{p-2k+1}}{p-2k+2}.$$

This last expression is easily seen to be positive for $2<s<3$, and the proof is now complete.

A Note by Otto Dunkel. This proof consists of two parts both of which are interesting, but it appears that the second part concerning the difference formula with the unit difference two (6) would suffice. For it is easily seen that (6) is true for p odd or even. Moreover the last term on the right in this formula is positive for the intervals (1) and (2) for integral values of j from 1 to ∞. Hence $\Delta f(s, p)$ is positive in the respective intervals; and, since $f(s, 1)$ and $f(s, 2)$ are positive in the same intervals, it follows that $f(s, p)$ is also positive in the same intervals.

3522. [1932, 45]. *Proposed by A. Galbraith, Ash Grove, Mo.*

To trisect any angle, approximately, with straight edge and compasses only take the vertex of the given angle as a center and with any radius describe the arc AC. Divide the chord AC into six equal parts, the middle point being B and the other two points between B and C being F and G in order B, F, G, C. On AC produced, locate the points H and K such that $CH=\frac{1}{6}AC$ and $HK=\frac{1}{2}CH$. At F erect a perpendicular to AC intersecting the arc AC in E. With G as a center and a radius EG, describe an arc intersecting the chord AC in M. With K as a center and radius EK describe an arc intersecting AC in N. Locate point P, the mid-point of MN. With E as a center and radius PH, describe an arc intersecting AC in L. With L as a center and a radius EL describe an arc intersecting AC in O. Then CO will be the chord of the trisected arc AC.

A Note by the Editors: This is the most accurate method of trisecting any angle approximately by means of a straight edge and compass only that has ever come under our observation.

We are publishing it for the double purpose of acquainting our readers with this rather complex though very accurate method and to have some of them determine for what angles the method is absolutely exact. We suggest that those who are interested, take a ruler and compasses and follow out the construction. An ocular proof is thus seen to be very convincing to the non-mathematical mind.

A Note by Otto Dunkel. The point L is located on the segment CK rather than on AC. The direct computation of the angle subtended by the chord CO is quite laborious, and the ordinary five place logarithm tables may not be sufficiently accurate to determine the error. Also an exact expression for the error ϵ in terms of the trisected angle θ is not easily obtained. An approximate expression for the error may be obtained by series developments which gives a fairly

good idea of the accuracy of the method. The first two non-vanishing terms are given by

$$\epsilon = \frac{8 \sin^5 \frac{1}{2}\theta}{45927 \cos \theta/6}\left[17 - \frac{40955}{3969}\sin^2 \tfrac{1}{2}\theta \cdots\right], \tag{1}$$

where ϵ is the excess of $\theta/3$ over the approximate trisection expressed in radians. For $\theta = 90°$, (1) gives $\epsilon = 1'\ 17.9''$, whereas the direct computation gives $1'\ 22''$. For $\theta = 45°$, (1) gives $\epsilon = 4.67''$, whereas the direct computation gives $4.8''$. For very small angles, an approximate expression is

$$\epsilon = \frac{\theta^5}{10806.+}. \tag{2}$$

For angles not greater than 45° a simpler and more accurate method is as follows:

If BAC is the angle to be trisected, describe a circle with the vertex A as center and with any convenient radius, AC, cutting AB in the diameter DB. Produce AD to N and M so that $AN = 2AC$ and $AM = DC$. Determine the point P so that it divides the segment MN in the ratio 5:4. Let PC cut the circle again in Q: determine on AN the point P' so that $QP' = QA$. Then the angle $AP'C$ is approximately one third the angle BAC.

The first part of the construction giving the approximate trisection angle APC was given by Wedderburn in problem 2972 [1922, 224], while the rest was described by Kennedy in his paper, *Angle Division*, in this Monthly [1932, 478]. The approximation exceeds the exact trisection, and the excess for very small angles is approximately

$$\epsilon = \frac{\theta^7}{47239.2}. \tag{3}$$

For angles near 90° this method may not be as accurate as that of the problem, since then the operation for determining P' is not so effective in reducing the error of the first part. For $\theta = 90°$ this operation reduces the error of the first part by only $\frac{1}{2}$: thus the error by the first part is $6'\ 30.6''$, and the last step reduces the error to $3'\ 15.3''$. However, it is not necessary to consider angles between 45° and 90°. For $\theta = 45°$, the error by the complete method is about $0.94''$, and for very small angles (3) shows that it is much more accurate than the problem method. Moreover, this method has the additional advantage that it can be made of any degree of accuracy by repeating the process for finding P'.

A construction of about the same order of accuracy as that of the problem is as follows:

If the angle to be trisected is BAC, take any convenient length AC on that side and produce BA to D and F so that $AD = DF = AC$. Draw DC and take on this segment $DE = DF$. On AF lay off $AM = DC$ and $AN = FE$. On MN take

the point P so that $MP:PN=20:7$; then angle APC is approximately one third of angle BAC.

For $\theta=45°$ the error is about $4''$: for angles not greater than $45°$ the error is given approximately by

$$\epsilon = 67.775 \sin\theta/3 \sin^4\theta/24. \tag{4}$$

For very small angles

$$\epsilon = \frac{\theta^5}{15996.+}. \tag{5}$$

The approximation is too small. The error is likely to be less than the result given by (4).

Approximate methods for trisecting an angle, or for finding any part of a given angle, may be obtained in a variety of ways. One method is to determine the constants in

$$\tan\delta = \frac{a\sin\theta + b\sin\frac{1}{2}\theta}{c\cos\theta + d\cos\frac{1}{2}\theta + e}, \tag{6}$$

so that for very small angles δ is very nearly $\theta/3$. It will be seen at a glance that we must have $2(c+d+e)=6a+3b$, and this makes the error of the third order in θ. If greater accuracy is desired, other conditions are imposed so that the error is of higher order in θ with as small a coefficient of the principal part of the error as possible. The resulting formula for very accurate constructions may not turn out to be very convenient for construction, as may be observed in the last entry of the table below. In this table the first three rows give very simple constructions, but they are not nearly so accurate as those given by the lower rows.

	a	b	c	d	e
1	1	0	1	0	2
2	1	0	1	2	0
3	1	0	1	1	1
4	9	0	9	8	10
5	1	16	1	16	10
6	7	40	7	44	30

The first four in the table have been mentioned in this MONTHLY. The errors for very small angles for the last two are approximately

$$\epsilon_5 = \frac{\theta^5}{34992}, \quad \epsilon_6 = \frac{\theta^7}{2519424}.$$

The method may be varied by introducing sines in the numerator and cosines in the denominator of $\theta/2^2$, $\theta/2^3$, etc. The construction above to which (4) applies will be found to employ $\theta/4$.

3541. [1932, 175]. *Proposed by J. B. Reynolds, Lehigh University.*

A rod of length $2a$ standing on a rough level plane of coefficient of friction μ falls from a vertical position. At what angle with the vertical will the rod begin to slip at the bottom?

Solution by V. F. Ivanoff, San Francisco, California.

The equations of the motion of the rod before it has begun to slip, are:

$$Mg\cos\theta - R_n = Ma\omega^2, \tag{1}$$

$$Mg\sin\theta - R_t = Ma\alpha, \tag{2}$$

$$Mg\sin\theta \cdot a = I\alpha, \tag{3}$$

where

M is the mass of the rod;

$I=(4/3(Ma^2$, the moment of inertia of the rod about its center of rotation;

α, the angular acceleration;

ω, the angular velocity;

θ, the angle between the rod and vertical;

R_t, the reaction normal to the rod; and

R_n, the reaction directed along the rod.

There is a relation between θ, ω, and α, namely: $d\theta/d\omega=\omega/\alpha$; whence, using (3) we get, $I\omega d\omega=aMg\sin\theta d\theta$, and taking into account that when $\theta=0$, ω also is 0, we get by integration,

$$\omega^2 = 2aMg(1-\cos\theta)/I = 3g(1-\cos\theta)/2a. \tag{4}$$

Now, from (1), (2), (3), and (4) we have

$$R_n = Mg\cos\theta - Ma\omega^2 = \tfrac{1}{2}Mg(5\cos\theta - 3)$$

and

$$R_t = Mg\sin\theta - Ma\alpha = \tfrac{1}{4}Mg\sin\theta.$$

The vector sum of these reactions has the components in the vertical and horizontal directions as follows:

$$R_v = R_t\sin\theta + R_n\cos\theta = \tfrac{1}{4}Mg(9\cos^2\theta - 6\cos\theta + 1)$$

$$R_h = R_t\cos\theta - R_n\sin\theta = \tfrac{3}{4}Mg\sin\theta(2 - 3\cos\theta).$$

The rod begins to slip when the horizontal force R_h becomes equal to μR_v, taken with the same sign as R_h. If $\theta>\cos^{-1}\frac{2}{3}$, R_h is positive and we have

$$\mu(9\cos^2\theta - 6\cos\theta + 1) = 3\sin\theta(2 - 3\cos\theta):$$

whence

$$\mu = \frac{3 \sin\theta(2 - 3\cos\theta)}{-3\cos\theta(2 - 3\cos\theta) - 1};$$

that is, μ is negative, which is inadmissible.

Again, if $\theta < \cos^{-1}\frac{2}{3}$, R_A is negative and we have

$$\mu(9\cos^2\theta - 6\cos\theta + 1) = -3\sin\theta(2 - 3\cos\theta),$$

or

$$3(\mu\cos\theta - \sin\theta)(2 - 3\cos\theta) = \mu.$$

The solution of this equation (quartic in trigonometric functions) gives the desired angle.

Also solved by William Hoover.

A Note by the Proposer. The last equation above is of the form $f(\theta) = \mu$. In order for this equation to be satisfied we must have

$$\tan^{-1}\mu < \theta < \cos^{-1}\tfrac{2}{3},$$

since

$$f(\theta) = 3\cos\theta(\mu - \tan\theta)(2 - 3\cos\theta) = \mu$$

can be satisfied only if both parentheses are negative. This condition will always obtain since $\tan(\cos^{-1}\frac{2}{3}) = 5^{1/2}/2 > 1 > \mu$.

3559. [1932, 359]. *Proposed by G. A. Yanosik, New York University.*

Variable circles are drawn having any point on a central conic and one of its foci as ends of a diameter. Prove that the envelope of these circles is the auxiliary circle.

Solution by E. F. Allen, Stillwater, Oklahoma.

Assume the equation of the conic to be

$$\frac{x^2}{a} + \frac{y^2}{a(1 - e^2)} = 1, \text{ where } a > 0.$$

The equation of the circle upon the line joining the focus, $F(e\sqrt{a}, 0)$ and the point $P(x, y)$ of the conic, as a diameter is

$$\xi^2 + \eta^2 - \xi(x + \sqrt{a}e) + \sqrt{a}ex - \eta y = 0,$$

where x is the parameter and y is a function of x, obtained from the equation of the conic. The envelope of this family of circles, obtained in the usual way, is represented by the equation

$$(\xi^2 + \eta^2 - \xi\sqrt{a}e)^2 - [a(\xi - \sqrt{a}e)^2 + a(1 - e^2)\eta^2] = 0.$$

The left member of this equation being expanded and then factored gives

$$(\xi^2 + \eta^2 - a)(\xi^2 - 2\sqrt{a}e\xi + ae^2 + \eta^2) = 0.$$

Hence, we have

$$\xi^2 + \eta^2 - a = 0 \text{ and } \xi^2 - 2\sqrt{a}e\xi + ae^2 + \eta^2 = 0,$$

the first being the equation of the auxiliary circle and the second the equation of the focus, which was used as the end of the diameter.

Also solved by W. B. Campbell, A. Pelletier, C. A. Rupp, F. Underwood, Paul Wernicke and the proposer.

A Note by Otto Dunkel. If Q and Q' are two neighboring points on the curve (here an ellipse), the two circles with diameters FQ and FQ' intersect in P' so that FP' is perpendicular to the chord QQ' of the curve. Hence the point of contact P of the circle FQ with its envelope is the foot of the perpendicular from F to the tangent at Q to the curve. The envelope is thus the pedal of the curve with respect to F.

Let O and F' be the center and the other focus of the ellipse, and let $F'Q$ produced meet FP produced in G. Since for the ellipse the tangent QP bisects the angle FQG, we have $F'G = F'Q + FQ = 2a$. It now follows that $OP = a$, and hence the envelope is the auxiliary circle of the ellipse. A similar analysis applies to the hyperbola and to the parabola. See the note [1933, 58] to the solution of 3535 for the converse theorems.

3560. [1932, 359]. *Proposed by Frank Morley, Johns Hopkins University.*

In a Euclidean space, perpendiculars from the vertices of a regular tetrahedron to a plane meet the plane at the points represented by the complex numbers x_i, $i = 1, 2, 3, 4$. Show that the four points obey the relation

$$\sum^{6}(x_i - x_j)^2 = 0.$$

Solution by F. Underwood, University College, Nottingham.

In a regular tetrahedron $ABCD$, pairs of opposite edges (such as AB, CD) may be taken as diagonals of opposite faces of a cube which is the enveloping (or surrounding) parallelepiped for the tetrahedron.

If the edge of this cube is $2a$, by taking rectangular axes through its centre (each axis being perpendicular to two faces), the co-ordinates of the vertices of the tetrahedron can be taken as

$$A(a, -a, a);\ B(-a, a, a);\ C(a, a, -a);\ D(-a, -a, -a).$$

If A_1, B_1, C_1, D_1, are the feet of the perpendiculars from A, B, C, D respectively to the plane $lx + my + nz = p$, where l, m, n are actual direction-cosines, we find

$$A_1 \text{ is } \{a + lp - al(l - m + n),\ -a + mp - am(l - m + n),\ a + np - an(l - m + n)\};$$

$$B_1 \text{ is } \{-a+lp-al(-l+m+n),\ a+mp-am(-l+m+n),$$
$$a+np-an(-l+m+n)\};$$
$$C_1 \text{ is } \{a+lp-al(l+m-n),\ a+mp-am(l+m-n),$$
$$-a+np-an(l+m-n)\};$$
$$D_1 \text{ is } \{-a+lp+al(l+m+n),\ -a+mp+am(l+m+n),$$
$$-a+np+an(l+m+n)\}.$$

Taking a new origin O_1 at (lp, mp, np), the foot of the perpendicular from O to the plane, and rectangular axes O_1Y, O_1Z in this plane, we can write

$$Y=\lambda_1(x-lp)+\mu_1(y-mp)+\nu_1(z-np),$$
$$Z=\lambda_2(x-lp)+\mu_2(y-mp)+\nu_2(z-np),$$

where λ_1, μ_1, ν_1 and λ_2, μ_2, ν_2 are subject to the restrictions

$$l\lambda_1+m\mu_1+n\nu_1=l\lambda_2+m\mu_2+n\nu_2=0$$
$$\lambda_1\lambda_2+\mu_1\mu_2+\nu_1\nu_2=0$$
$$\lambda_1^2+\mu_1^2+\nu_1^2=\lambda_2^2+\mu_2^2+\nu_2^2=1,$$

but are otherwise arbitrary.
Then

$$\frac{l}{\mu_1\nu_2-\mu_2\nu_1}=\frac{m}{\nu_1\lambda_2-\nu_2\lambda_1}=\frac{n}{\lambda_1\mu_2-\lambda_2\mu_1}=1.$$

To simplify the algebra in the remaining work it is convenient to take $\nu_1=\lambda_2=0$. Then $l=\mu_1\nu_2$, $m=-\lambda_1\nu_2$, $n=\lambda_1\mu_2$. Also $\mu_1\mu_2=0$; $\lambda_1^2+\mu_1^2=\mu_2^2+\nu_2^2=1$. The complex numbers for the points A_1, B_1, C_1, D_1 can be taken as $x_n=Y_n+iZ_n$, and if the suffixes 1, 2, 3, 4 are used in this order, it is easily found that

$$x_1/a=\lambda_1-\mu_1+i(\nu_2-\mu_2).$$
$$x_2/a=\mu_1-\lambda_1+i(\mu_2+\nu_2).$$
$$x_3/a=\lambda_1+\mu_1+i(\mu_2-\nu_2).$$
$$x_4/a=-\lambda_1-\mu_1-i(\mu_2+\nu_2).$$

Hence if the required sum is $S=\sum^6(x_i-x_j)^2$,

$$S/(4a^2)=\{(\lambda_1-\mu_1)-i\mu_2\}^2+\{\mu_1+i(\mu_2-\nu_2)\}^2+(\lambda_1+i\nu_2)^2$$
$$+(-\lambda_1+i\nu_2)^2+\{\mu_1+i(\mu_2+\nu_2)\}^2+\{(\lambda_1+\mu_1)+i\mu_2\}^2$$
$$=4(\lambda_1^2+\mu_1^2-\mu_2^2-\nu_2^2+i\mu_1\mu_2)=0.$$

Also solved by J. M. Feld and the proposer.

A Note by Otto Dunkel. The proof may be put in the following form: Let X', Y', Z' denote the system of coordinate axes as above. The projections of the four points on any one of the coordinate planes, say $X'Y'$, form a square of

side $2a$; and, if its vertices be denoted by $X_j' + Y_j' i$, it is verified by inspection of the sides and diagonals of the square that the theorem is true for these special cases. Hence in this case we have

$$(1)\quad \sum^{6}(X_j' - X_k')^2 = \sum^{6}(Y_j' - Y_k')^2 = 16a^2, \quad \sum^{6}(X_j' - X_k')(Y_j' - Y_k') = 0,$$

and there are two similar sets of equations for the remaining two planes.

Consider now any new system of rectangular axes X, Y, Z with the same origin. Then

$$(2)\quad \begin{aligned} X &= l_1X' + l_2Y' + l_3Z', \\ Y &= m_1X' + m_2Y' + m_3Z', \end{aligned}$$

$$x = X + Yi = (l_1 + m_1 i)X' + (l_2 + m_2 i)Y' + (l_3 + m_3 i)Z', \; i = \sqrt{-1}.$$

Hence

$$\begin{aligned} \sum^{6}(x_j - x_k)^2 &= (l_1 + m_1 i)^2 \sum^{6}(X_j' - X_k')^2 \\ &\quad + \cdots 2(l_1 + m_1 i)(l_2 + m_2 i) \sum^{6}(X_j' - X_k')(Y_j' - Y_k') \cdots \\ &= 16a^2[\, \textstyle\sum l^2 + 2i \sum lm - \sum m^2] = 0, \end{aligned} \qquad (3)$$

where in the first two lines two sums in each have been indicated by the dots.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus Ohio.

THE SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION OF AMERICA

The Mathematical Association of American will hold its summer meeting in Chicago during the week of June 19, 1933 in conjunction with the summer meetings of the American Association for the Advancement of Science and the American Mathematical Society. The American Association will have a program for two weeks, beginning June 19, in connection with the Century of Progress International Exposition, with emphasis on pure science the first week and on applied science the second week. The mathematical meetings will be held for the most part at the University of Chicago. There will be addresses by two foreign mathematicians, Professor Leopold Fejér of the University of Budapest and Professor Tullio Levi-Civita of the University of Rome, and symposia in which foreign and American mathematicians will participate. The full program of the Mathematical Association will be sent to our members in ample season.

Accommodations for mathematicians and their guests will be furnished at

Judson Court, college residence halls for men at the University of Chicago, located at 60th Street and Ellis Avenue. The charge for room will be $2.00 per day per person, and the charge for meals will be $2.00 per day. A weekly rate of $25 for room and board has been set. Suitable arrangements for families can be made at the residence halls. It is anticipated that many of those attending the mathematical meeting will wish to remain for a few days after its conclusion for a more thorough examination of the many interests of the Century of Progress Exposition. For those persons who wish to remain longer, arrangements have been made enabling them to continue residence at Judson Court for a short time. Requests for residence hall or any other accommodations, and for information concerning the Exposition, should be sent to Professor H. S. Everett, University of Chicago, Chicago, Ill.

Plans are being made for the Conference of Engineers at Chicago during the week of June 25–30. On the evening of June 25 the International Union of Pure and Applied Physics will have a joint session with section M, of the American Association for the Advancement of Science. At this meeting Professor R. A. Millikan will speak on the topic "Applications of physics to engineering."

The Danish government reports the approval of a sum of 50,000 kroner for exhibits at the Century of Progress Exposition at Chicago. These exhibits will consist of originals and replicas of the apparatus used by Tycho Brahe, Ole Römer, Oersted, Niels Bohr, and others.

Dr. William Bowie of the U. S. Coast and Geodetic Survey has been awarded the Charles Lagrange prize by the Royal Academy of Belgium in recognition of his having affected the unification of the triangulation systems of the United States, Canada, and Mexico.

Dr. Ernest Brown, professor of mathematics at Yale, gave the second Arthur lecture at the Smithsonian Institution on January 25. His subject was "Gravitation in the solar system."

Professor Edward Kasner of Columbia University gave three public lectures in January and February before the Peoples Institute. First, Numbers and infinity; second, Spaces and dimensionality; third, Geometry and physics.

Professor Warren Weaver, chairman of the department of mathematics of the University of Wisconsin, has resigned his post to accept permanent charge of the Natural Science Division of the Rockefeller Foundation, a post which he assumed temporarily during a year's leave of absence. His resignation has been accepted by the regents and Professor M. H. Ingraham has been named chairman of the department.

Professor Jesse Douglas of the Massachusetts Institute of Technology has been granted leave of absence for the second semester of the present academic year.

The following courses in mathematics are announced for the summer of 1933:

University of Chicago, first term, June 19 to July 21; second term, July 24 to August 25. In addition to Calculus I and Elementary differential equations, the following advanced courses will be offered: By Professor G. A. Bliss: Calculus of variations; Multiple integrals. By Professor A. C. Lunn: Vector analysis; Physical applications of the theory of groups. By Professor M. I. Logsdon: Analytic projective geometry; Algebraic geometry I. By Professor L. M. Graves: Introduction to higher algebra; Integral equations. By Professor A. A. Albert: Elementary theory of equations; Topics in modern algebraic theories. By Professor Walter Bartky: Special perturbations; Modern theories of analytic differential equations I.

Columbia University, July 10 to August 18. In addition to courses in trigonometry, solid geometry, analytic geometry, calculus, and methods of teaching secondary mathematics, the following advanced courses are offered: By Professor E. Kasner: General concepts of mathematics; Geometric transformations and continuous groups. By Professor W. B. Fite: Differential equations; Introduction to higher algebra. By Professor P. A. Smith: Theory of functions of a real variable.

Cornell University, July 8 to August 18. In addition to the usual elementary work, the following advanced courses will be offered: By Professor Hutchinson: Modern algebra. By Professor Virgil Snyder: Projective geometry. By Professor W. A. Hurwitz: Advanced calculus. By Professor W. B. Carver: Advanced analytic geometry. By Professor D. C. Gillespie: Elementary differential equations. Reading and research work will be directed by Professors J. I. Hutchinson, Virgil Snyder, F. R. Sharpe, W. A. Hurwitz, W. B. Carver, D. C. Gillespie, and C. F. Craig.

The George Washington University, nine weeks term, June 12 to August 11. In addition to the regular courses in plane analytic geometry, differential and integral calculus, the following advanced courses will be offered: By Professor F. E. Johnston: Advanced algebra. By Professor E. W. Woolard: Theory of potential.

University of Illinois, June 19 to August 12. In addition to the usual courses in college algebra, trigonometry, analytic geometry, and calculus, the following advanced courses are offered: By Professor R. D. Carmichael: Analysis. By Professor J. B. Shaw: Algebra; Introduction to higher algebra. By Associate Professor E. B. Lytle: Teacher's course; History of mathematics. By Assistant Professor Harry Levy: Geometry; Introduction to higher geometry. By Dr. Ketchum: Introduction to higher analysis.

University of Iowa, First term, June 12 to July 20. In addition to courses in college algebra, trigonometry, analytic geometry, and calculus, the following

subjects are offered: By Miss Ruth Lane: Subject matter and teaching of mathematics. By Professor Chittenden: Differential equations; Functions of real variables; Seminar in analysis. By Associate Professor Wylie: Mathematics of finance, astronomy, meteors. By Associate Professor Woods: Theory of equations; Projective geometry; Theory of algebraic invariants. By Dr. Craig: Topics in Advanced statistics. By the Staff: Reading and research.

Second term, July 24 to August 24. By Professor Reilly: Elementary finite difference equations; Introduction to Laplace's and Poisson's equations; Seminar in interpolation. By Assistant Professor Ward: Infinite series; the Calculus of variations. By Dr. Conkwright: Differential equations; Group theory. By the Staff: Reading and research.

Johns Hopkins University, June 26 to August 5. By Professor F. D. Murnaghan: College algebra; Differential and integral calculus; Functions of a complex variable.

University of Kansas, June 14 to August 9. In addition to the usual elementary courses, the following advanced courses are offered: By Professor Mitchell: Theory of numbers; History of mathematics; Teachers' course; Seminar. By Professor Smith: Modern synthetic geometry; Seminar. By Professor Wheeler: Differential equations; Mathematical theory of statistics.

University of Kentucky, First term. By Professor H. H. Downing: Theory of equations. By Dean P. P. Boyd: Algebraic plane curves. By Professor F. E. LeStourgeon: Calculus of variations.

Second term. By Professor C. G. Latimer: Theory of numbers; and another course to be chosen. By Professor L. W. Cohen: Theory of functions of a real variable.

University of Maine, July 5 to August 12. In addition to the usual elementary work, the following advanced courses are offered: By Associate Professor Bryan: History of mathematics; Teachers' course. By Associate Professor Jordan: Practical astronomy. By Professor Willard: Differential equations, or other graduate courses by arrangement.

Massachusetts Institute of Technology, First period, June 13 to July 25, Calculus and differential equations, covering the prescribed work of the first two years; Course in theoretical aeronautics; Advanced calculus. July 5 to July 25. Course in aeronautics continued. Second period, July 26 to September 6. Courses in the first period repeated; Vector analysis. August 7 to September 9. Courses in algebra, solid geometry and trigonometry, in preparation for fall entrance examinations in these subjects.

University of Michigan, June 26 to August 18. In addition to courses in algebra, trigonometry, analytic geometry, elementary calculus, statistics, and finance, the following advanced courses will be offered: By Professor John W. Bradshaw: Topics in calculus; The figures of solid geometry. By Professor V. C.

Poor: Differential equations; Applied mathematics—Engineering problems. By Professor Peter Field: Higher algebra; Vector analysis. By Professor Cecil C. Craig: Theory of probability; Graphical methods; Mathematical theory of statistics. By Professor H. C. Carver: Finite differences. By Professor R. V. Churchill: Solid analytic geometry. By Professor L. J. Rouse: Solid analytic geometry; Advanced calculus. By Professor L. A. Hopkins: Analytic mechanics. By Professor W. L. Ayres: Advanced calculus. By Professor C. J. Coe: Infinite series, with special reference to Fourier series. By Professor L. C. Karpinski: Teaching of geometry; History of mathematics. By Professor G. Y. Rainich: Higher geometry; Differential geometry; Mathematics of relativity. By Professor R. L. Wilder: Introduction to the foundations of mathematics; Studies in the foundations and point sets. By Professor T. H. Hildebrandt: Theory of functions of a real variable; Partial differential equations. By Professors Hildebrandt, Rainich, and others: Seminar in pure mathematics.

University of Minnesota, first term, June 20 to July 29. In addition to the usual elementary work the following courses will be offered: By Professor Dunham Jackson: Vector analysis. By Associate Professor Anthony L. Underhill: Differential equations. By Professors Elizabeth Carlson, Jackson, and Underhill: Reading in advanced mathematics. By Assistant Professor Gladys Gibbens: Famous problems in geometry. Second term, July 31 to September 2. No courses will be offered beyond the sophomore level.

Northwestern University, June 19 to August 12. In addition to courses in trigonometry, college algebra, analytic geometry, differential and integral calculus, the following advanced courses will be offered: By Professor F. E. Wood: Determinants and theory of equations. By Dr. N. E. Rutt: Advanced calculus. By Professor H. S. Wall: Definite integrals.

Ohio State University, Summer Quarter, June 20 to September 1. In addition to elementary courses in college algebra, analytics, and calculus, the following advanced courses are offered: By Professor S. E. Rasor: Functions of a complex variable; Fourier Series. By Professor Tibor Radó: Vector analysis; Differential geometry. By Assistant Professor Grace M. Bareis: Theory of equations; Introduction to higher geometry. By Dr. L. E. Bush: Introduction to higher algebra.

University of Pittsburgh, July 3 to August 11. In addition to the undergraduate courses the following more advanced courses will be offered: By Professor F. A. Foraker: Advanced calculus; Analytical projective geometry. By Professor J. S. Taylor: Vector analysis; Modern algebraic theories. By Professor M. M. Culver: Differential equations.

Stanford University, June 22 to September 2. In addition to the usual courses in calculus and differential equations, the following courses will be offered: By Professor W. A. Manning of Stanford University: Higher geometry; Fuchsian

groups; Theory of functions of a complex variable. By Professor M. H. Stone of Yale University: Some fundamental concepts of mathematics. By Professor G. Polys of Technical School, Zürich, Switzerland: Lectures on singularities of power series.

University of Southern California, first term, June 19 to July 28. In addition to the usual elementary courses, the following advanced courses are offered: By Professor L. D. Ames: History of mathematics; Selected topics—algebra; Theory of probability and statistics. By Associate Professor D. V. Steed: Projective geometry. Second term, July 31 to September 1. By Professor L. E. Gurney: Theory of equations and determinants; Mathematical astronomy; Seminar (subject to be announced during first term).

Syracuse University, July 6 to August 11. In addition to the regular courses in plane and solid geometry, intermediate and advanced algebra, trigonometry, analytic geometry, and differential and integral calculus, the following courses will be offered: By Professor I. S. Carroll: Teaching of mathematics. By Professor F. F. Decker: Elementary theory of numbers or Fundamental concepts in mathematics. By Professor A. D. Campbell: Advanced plane analytic geometry or Elementary solid analytic geometry.

University of Vermont. By Professor Bullard: Differential calculus; Analytical geometry. By Professor Butterfield: Descriptive astronomy; History of mathematics; Plane trigonometry. By Professor Millington: Elementary algebra; College algebra; Solid geometry. By Professor Swift: Integral calculus; The teaching of plane geometry.

University of Wisconsin, six weeks session, June 26 to August 4. By Professor H. P. Evans: Differential and integral calculus; Topics in the theory of probability. By Dr. M. L. Hartung: The content of secondary mathematics; The teaching of mathematics. By Professor M. H. Ingraham: Mathematics of educational statistics; The elementary properties of number systems. By Professor H. W. March: Analytical geometry; Dimensional analysis. By Professor I. S. Sokolnikoff: Advanced calculus; Differential equations; Conformal representation. By Professor E. B. Skinner: Differential geometry; Theory of equations; Theory of numbers. Special nine weeks session for graduates, June 26 to August 25. These courses may be taken for six weeks. By Professor M. H. Ingraham: Advanced analytic theory of equations. By Professor H. W. March: Harmonic analysis.

MATHEMATICAL ASSOCIATION OF AMERICA

The following twenty-four persons and one institution have been elected to membership in the Association on applications duly certified:

To Individual Membership

M. W. AYLOR, M.S. (Virginia) Instr., Univ. of Virginia, University, Va.

FANNIE W. BOYCE, A.M. (Wisconsin) Asst. Prof., Wheaton Coll., Wheaton, Ill.

O. Z. BURDICK, M.S. (Illinois) Engineer with firm of C. W. Hills, Patent Lawyers, Chicago, Ill.

SISTER M. CLAUDETTE, A.M. (Minnesota) Head of Dept., Coll. of St. Benedict, St. Joseph, Minn.

F. F. CRANDELL, A.B. (Wooster) Grad. student, Ohio State Univ., Columbus, Ohio

R. D. DALAL, Stock Broker, Bombay, India

D. C. DEARBORN, A.B. (Hastings Coll.) Asst. Instr., Univ. of Nebraska, Lincoln, Nebr.

H. T. ENGSTROM, Ph.D. (Yale) Asst. Prof., Yale University, New Haven, Conn.

NICHOLAS FATTU, M.S. (Idaho) Instr., Itasca Jr. Coll., Coleraine, Minn.

J. A. FLEMING, B.S. (Cincinnati) Acting Director, Dept. of Terrestrial Magnetism, Carnegie Institution of Washington, Washington, D. C.

BEATRICE L. HAGEN, Ph.D. (Chicago) Instr., Pennsylvania State Coll., State College, Pa.

I. L. HEBEL, B.S. (Colorado) Asst. Prof., Colorado School of Mines, Golden, Colo.

B. O. KOOPMAN, Ph.D. (Harvard) Asst. Prof., Columbia Univ., New York, N. Y.

ABE LASSER, Student, Armour Inst. of Tech., Chicago, Ill.

F. S. NOWLAN, Ph.D. (Chicago) Prof., Univ. of British Columbia, Vancouver, B. C.

W. T. PUCKETT, JR., M.S. (Virginia) Grad. student, Univ. of Virginia, University, Va.

S. R. TARAZI, A.B. (Amer. Univ. of Cairo, Egypt) Teacher, Friends' Boys' School, Ram Allah, Palestine

R. B. THOMPSON, A.B. (Hastings Coll.) Asst. Instr., Univ. of Nebraska, Lincoln, Nebr.

H. F. TODD, Teacher, Jr. High School, Leewood, W. Va.

JOHN VON NEUMANN, Ph.D. (Budapest), Ing. Chem. (Zurich) Prof., Inst. for Advanced Study, Princeton, N. J. (Until Sept. 1933, Budapest, Hungary)

T. L. WADE, JR., M.S. (Virginia) Asst. Instr., Univ. of Virginia, University, Va.

H. E. WAHLERT, A.M. (Princeton) Instr., Washington Square Coll., New York Univ., New York, N. Y.

H. A. WRIGHT, Ph.D. (New York Univ.) Acting Head of Dept., Transylvania Coll., Lexington, Ky.

J. H. ZANT, A.M. (Columbia) Oklahoma A & M Coll., Stillwater, Okla.

To Institutional Membership

ST. MARY'S COLLEGE, Notre Dame, Ind.

The Trustees have approved the organization and by-laws of the Oklahoma Section.

W. D. CAIRNS, *Secretary*

THE NINETEENTH ANNUAL MEETING OF THE KANSAS SECTION

The nineteenth annual meeting of the Kansas Section of the Mathematical Association of America was held in Topeka, at the High School Building, on Saturday, February 11, 1933. In the morning the session was a joint meeting with the Kansas Association of Mathematics Teachers; Professor O. J. Peterson, Chairman of the Section, presided at both sessions.